THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring**

Problem Set 7

Haykin section 3.6
Web notes on convexity
Web notes on quantization

1. **Quantization:** Show that a nonuniform quantizer with sufficiently small bin sizes Δ_i, has mean-square quantization error of approximately $\frac{1}{12}\sum_i \Delta_i^2 p_i$, where p_i is the probability that the input signal amplitude lies within the ith interval.

   HINT: You may assume that for Δ_i sufficiently small, you have an approximately uniform distribution given that x lies in the ith bin.

   SOLUTION: *First we invoke Lloyd-Max. Since we assume the conditional distribution in any bin is approximately uniform, we must have the associated level q_i as the midpoint of the bin. If the bin starts at x_{i-1}, it must end at $x_i = x_{i-1} + \Delta_i$. Then we must have $q_i = x_{i-1} + \Delta_i/2$ (or $q_i = x_i - \Delta_i/2$). Let's talk only in terms of the q_i. Then we know that the ith bin is $[q_i - \Delta_i/2, q_i + \Delta_i/2]$.*

   Now, GIVEN you're in the ith bin, the probability distribution is uniform with amplitude $1/\Delta_i$. Therefore the mean square error in a given bin is $e_i^2 = E[(X - q_i)^2|X \in \textit{bin}_i]$. We define

   $$p_i = \int_{x_{i-1}}^{x_i} f_X(x)dx$$

   so that

   $$e_i^2 = \int_{x_{i-1}}^{x_i} (X - q_i)^2 \frac{1}{\Delta_i} dx = \sigma^2_{X|X \in \textit{bin}_i}$$

   where the last equality comes from q_i being the conditional mean (via Lloyd-Max). We know the variance of a uniform distribution (or can calculate it) is $\Delta^2/12$ where Δ is the distribution width. So we have

   $$e_i^2 = \frac{\Delta_i^2}{12}$$

   Now, we have to calculate the average error and we have

   $$\sum_i p_i \frac{\Delta_i^2}{12}$$

   and we're done.

2. **Quantization Example:**

(a) A random waveform $x(t)$ has amplitude uniformly distributed over $[-1, 1]$. Please provide an optimal 4 bit quantizer for this signal. Show your choice satisfies the Lloyd-Max optimality conditions.

SOLUTION: *This is one of those questions which can be answered intuitively but then has to be followed up with some cold hard analysis. But it was meant to be a warmup. First, it's pretty clear you have to chop the interval $[-1, 1]$ up into 16 (4 bits) equal size bins. Let x_0 be the boundary between the first and second bin (with associated quantizer output q_0) and let x_{14} be the boundary which marks the beginning of the final quantizer bin (which has quantizer output q_{15}). Then we have (intuitively)*

$$x_j = -1 + \frac{1}{8}(j+1)$$

and

$$q_j = -\frac{15}{16} + \frac{1}{8}j$$

Testing for Lloyd-Max:

$$q_j = \int_{\textit{bin } j} x f_{X|\textit{bin } j}(x|\textit{bin } j)dx$$

and

$$x_j = \frac{q_j + q_{j+1}}{2}$$

CHECK: the conditional distribution given you're in a particular is still uniform over that bin (since the original distribution was uniform everywhere in the larger interval). Therefore, the conditional mean is the midway point of the bin. NOTE: for those bins on the end, you only look as far as ± 1 since the signal can't exist outside that range. So, the conditional mean for the first bin is -15/16 and for the last bin it's 15/16 and it's $-15/16 + j/8$ for all the others. This jibes with our result for q_j. Well, $q_j + q_{j+1} = -30/16 + j/4 + 1/8$ so that $x_j = -7/8 + j/8 = -1 + (j+1)/8$ which matches our initial result.

One could also have solved Lloyd-Max directly in this case. Regardless, we have uniform bin sizes of $\Delta = \frac{1}{8}$.

(b) Let $\hat{x}(0)$ be a quantized sample of the signal level $x(t_0)$. What is $E[\hat{x}(0) - x(t_0)]$? What is $E[(\hat{x}(0) - x(t_0))^2]$? What is the probability distribution on the random variable $e(0) = \hat{x}(0) - x(t_0)$? You must justify your results.

SOLUTION: *Let's consider the bin $[0, \frac{1}{8}]$ first. The distribution of the difference between the signal $x(t_0)$ and the quantized value is uniform over the bin (of width $1/8$). This result is the same for ANY bin.*

So, given that q_i is the midpoint of each bin, we must have

$$E[\hat{x}(0) - x(t_0)] = 0$$

The variance is the variance of a uniform distribution over the bin width: $\frac{\Delta^2}{12} = (1/8)^2/12$. Since the same argument applies to all other bins, and the probability that $x(0)$ is in any given been is the same, the total mean square error must be $\frac{\Delta^2}{12} = (1/8)^2/12 = \frac{1}{768}$.