THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring**

Problem Set 3

1. **Discovered Angle Modulation** A signal $s(t)$ is measured and found to be described by

$$s(t) = A\cos(2\pi f_a t + \alpha \sin 2\pi f_b t)$$

(a) We're later told that $s(t)$ is an angle modulated signal with sensitivity k_p. Using the standard angle modulation signal format found in your text, what is the information signal $m(t)$?

SOLUTION: *A phase-modulated signal $s(t)$ is a form of angle modulation in where the angle $\theta_i(t)$ is varied linearly with the message, thus this is described in time domain by*

$$s(t) = A\cos(2\pi f_c t + k_p m(t))$$

where

$$\theta(t) = 2\pi f_c t + k_p m(t)$$

Using the above equations, we see that

$$\theta(t) = 2\pi f_a t + \alpha \sin 2\pi f_b t$$

Hence,

$$f_c = f_a$$
$$k_p = \alpha$$
$$m(t) = \sin(2\pi f_b t)$$

with $f_m = f_b$

(b) Now, imagine that you're told "WHOOPS! I meant FREQUENCY modulation. with frequency sensitivity k_f." Again using the standard signal format described in your text, please provide the information signal $m(t)$ and the instantaneous frequency $f_i(t)$.

SOLUTION: *A frequency-modulated signal $s(t)$ is a form of angle modulation in which the instantenous frequency $f_i(t)$ is varied linearly with the message signal $m(t)$, and it is given by*

$$s(t) = A\cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau\right)$$

where the instantenous frequency is defined as

$$f_i(t) = fc + k_f m(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$$

Hence, given our signal $s(t)$ we find that

$$\theta_i(t) = 2\pi f_a t + \alpha \sin(2\pi f_b t)$$

and

$$\frac{d\theta_i(t)}{dt} = 2\pi f_a + \alpha \cos(2\pi f_b t)(2\pi f_b)$$

Therefore,

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt} = f_a + \alpha f_b \cos(2\pi f_b t)$$

$$f_c = f_a$$

$$k_f = \alpha f_b$$

$$m(t) = \cos(2\pi f_b t)$$

2. **Two Tone Madness:** Consider a message signal with two tones at frequencies f_a and f_b respectively, defined as

$$m(t) = A_m \cos(2\pi f_a t)\cos(2\pi f_b t)$$

   (a) Find the corresponding phase modulated and frequency modulated signals.

   SOLUTION: *For a PM signal, we have*

$$\begin{aligned} s(t) &= A\cos(2\pi f_c t + k_p m(t)) \\ &= A\cos(2\pi f_c t + k_p A_m \cos(2\pi f_a t)\cos(2\pi f_b t)) \\ &= A\cos\left(2\pi f_c t + \tfrac{k_p A_m}{2}\left[\cos(2\pi(f_a - f_b)t) + \cos(2\pi(f_a + f_b)t)\right]\right) \end{aligned}$$

   For a FM signal, we have

$$\begin{aligned} s(t) &= A\cos\left(2\pi f_c t + 2\pi k_f \textstyle\int_0^t m(\tau)d\tau\right) \\ &= A\cos\left(2\pi f_c t + \pi k_f A_m \textstyle\int_0^t [\cos(2\pi(f_a - f_b)\tau) + \cos(2\pi(f_a + f_b)\tau)]d\tau\right) \\ &= A\cos\left(2\pi f_c t + \tfrac{k_f A_m}{2}\left[\tfrac{\sin(2\pi(f_a - f_b)t)}{(f_a - f_b)} + \tfrac{\sin(2\pi(f_a + f_b)t)}{(f_a + f_b)}\right]\right) \end{aligned}$$

   (b) Find the narrowband FM (i.e NBFM) signal using the FM modulated signal obtained in the previous part.

   SOLUTION: *Given the FM signal obtained in Part (a), we can obtain a narrowband FM signal by using a modulation index β for which it is very small compared to one radian. Thus, the FM signal is*

$$s(t) = A\cos\left(2\pi f_c t + \beta\left[\frac{\sin(2\pi(f_a - f_b)t)}{(f_a - f_b)} + \frac{\sin(2\pi(f_a + f_b)t)}{(f_a + f_b)}\right]\right)$$

   where

$$\beta = \frac{A_m k_f}{2}$$

   We use trigonometric expansions to obtain

$$s(t) = A\cos(2\pi f_c t))\cos\left(\beta\left[\frac{\sin(2\pi(f_a - f_b)t)}{(f_a - f_b)} + \frac{\sin(2\pi(f_a + f_b)t)}{(f_a + f_b)}\right]\right) - A\sin(2\pi f_c t))\sin\left(\beta\left[\frac{\sin}{}\right.\right.$$

so the NBFM signal is approximately

$$s(t) = A\cos(2\pi f_c t) + A\beta\sin(2\pi f_c t)\left[\frac{\sin(2\pi(f_a - f_b)t)}{(f_a - f_b)} + \frac{\sin(2\pi(f_a + f_b)t)}{(f_a + f_b)}\right]$$

since for small $|x| \ll 1$ *we have* $\cos(x) \approx 1$ *and* $\sin(x) \approx x$

3. **Linear? Nonlinear?** Let $m_1(t)$ and $m_2(t)$ be two message signals, and let $s_1(t)$ and $s_2(t)$ be the corresponding modulated signals.

   (a) Carefully show that if the modulation is DSB-SC, SSB, or VSB, then

   $$m(t) = m_1(t) + m_2(t)$$

   will produce a modulated signal

   $$s(t) = s_1(t) + s_2(t)$$

   SOLUTION: *For* $m_1(t)$ *the modulated signal using DSB-SC is*

   $$s_1(t) = m_1(t)\cos(2\pi f_c t)$$

   Similarly, for $m_2(t)$ *using same modulation technique, we get*

   $$s_2(t) = m_2(t)\cos(2\pi f_c t)$$

   and for $m(t)$*, we have*

   $$s(t) = m(t)\cos(2\pi f_c t) = [m_1(t) + m_2(t)]\cos(2\pi f_c t)$$

   Hence,

   $$\begin{aligned} s(t) &= s_1(t) + s_2(t) \\ &= [m_1(t) + m_2(t)]\cos(2\pi f_c t) \end{aligned}$$

   Therefore, DSB-SC modulation is a linear modulation. Since SSB and VSB are simply linear operations (filtering) on this linear system, they are linear as well.

   (b) Show that if the modulation is PM or FM, then

   $$m(t) = m_1(t) + m_2(t)$$

   will not in general produce

   $$s(t) = s_1(t) + s_2(t)$$

   SOLUTION: *For PM, we have*

   $$s_1(t) = A\cos(2\pi f_c t + k_p m_1(t))$$

   and

   $$s_2(t) = A\cos(2\pi f_c t + k_p m_2(t))$$

   Thus,

   $$\begin{aligned} s(t) &= A\cos(2\pi f_c t + k_p m(t)) = A\cos(2\pi f_c t + k_p[m_1(t) + m_2(t)]) \\ &\neq s_1(t) + s_2(t) \end{aligned}$$

   So, we notice that the phase is linear in the program material $m_i(t)$ *but the overall signal is not. For FM, integration is a linear operation we use the results from PM to see that the instantaneous frequency and phase are linear but the actual modulated signal is not.*

4. **Lecture Redux** Suppose we have a modulated signal

$$s(t) = A\cos(2\pi f_c t + \beta sin(2\pi f_m t))$$

with $\beta << 1$ (i.e., narrowband FM/PM).

(a) Find the spectrum of this narrowband FM/PM signal.

SOLUTION: *Using the hint given in question 2 , we can approximate the NBFM signal to be*

$$\begin{aligned} s(t) &= A\cos(2\pi f_c t) + A\beta\sin(2\pi f_m t)\sin(2\pi f_c t) \\ &= A\cos(2\pi f_c t) + \tfrac{A\beta}{2}\left(\cos(2\pi(f_c - f_m)t) - \cos(2\pi(f_c + f_m)t)\right) \end{aligned}$$

Hence, the spectrum of s(t) is

$$S(f) = \frac{A}{2}[\delta(f - f_c) + \delta(f + f_c)] + \frac{A\beta}{4}[\delta(f - (f_c - f_m)) - \delta(f - (f_c + f_m))]$$

(b) Compare your previous result to the spectrum of an AM (suppressed carrier) signal

$$s(t) = A\sin(2\pi f_m t)\cos(2\pi f_c t)$$

Cite similarities and differences.

SOLUTION:

$$s(t) = \frac{A}{2}(\sin(2\pi(-f_c + f_m)t) + \sin(2\pi(f_c + f_m)t))$$

and

$$S(f) = \frac{A}{4j}[\delta(f - (-f_c + f_m)) + \delta(f - (f_c + f_m)) - 2\delta(f + (f_c + f_m))]$$

The AM signal is missing the large carrier present in the NBFM signal and the information is carried in phase with the carrier. IN the NBFM signal the information is $\pi/2$ *out of phase with the carrier. Thus, the envelope of the NBFM signal is more or less constant while (as its name implies) the AM signal varies in amplitude in direct proportion to the program material.*

5. **Phase Locked Loops:** Consider a phase locked loop whose sole purpose is to lock on to the incoming sinusoid $\cos(2\pi f_c t)$ and produce $\sin(2\pi f_c t)$ at the output of the VCO (voltage controlled oscillator). The incoming sinusoid is multiplied by the output of the VCO and the result is sent through a low pass filter whose output is multiplied by $-K$ ($K > 0$ a constant). The output of the multiplier, call it $\dot{\phi}(t)$, is in turn sent to the input of the VCO which produces $\sin\phi(t)$.

(a) Sketch a system diagram of the PLL and show it will produce the desired result if the gain K is large enough. You may assume that $\phi(t) \approx 2\pi f_c t$ and then make appropriate approximations.

SOLUTION: *The inputs to the multiplier are* $\cos(2\pi f_c t)$ *and* $\sin\phi(t)$. *We rewrite* $\phi(t) = 2\pi f_c t + e(t)$ *where* $e(t)$ *is assumed small. At the output of the multiplier we then* *have (after expanding the argument of the sine* $\cos^2(2\pi f_c t)\sin e(t)+\cos(2\pi f_c t)\sin(2\pi f_c t)$ *and the second term is lost when it passes through the low pass filter since it's HIGH frequency (*$\sin(4\pi f_c t)/2$ *to be exact). The first term has a D.C. component of* $1/2$ *(because of the cosine squared) and what pops out of the filter is* $\approx e(t)/2$ *since* $\sin x \approx x$ *for small* x*.*

We multiply this by $-K/2$ *and then have*

$$\dot{\phi}(t) = -Ke(t)/2 = -K(\phi(t) - 2\pi f_c t)/2$$

which we rewrite as

$$\dot{\phi}(t) + (K/2)\phi(t) = K\pi f_c t$$

This is a nice happy and stable first order differential equation whose homogeneous portion settles as $e^{-\frac{K}{2}t}$ *which for* K *large is FAST FAST FAST! So given the homogeneous solution dies out, we're left with the particular for* K *large of*

$$\phi(t) \approx 2\pi f_c t$$

which is exactly what we wanted in the first place! If you made it to here, you solved the problem and were done. The following is extra.

Now, above we assumed $e(t)$ *was small, but what if it's not small?!?! Specifically, what if* $e(t) = n\pi + \Delta(t)$ *where* $\Delta(t)$ *IS small but* n *can be ANY INTEGER!!!. All the same approximations hold, except that what pops out of the LPF is* $\pm\Delta(t)$*, not* $e(t)$*. We then have*

$$\dot{\phi}(t) = \pm K\Delta(t)/2 = \pm K(\phi(t) - 2\pi f_c t - n2\pi)/2$$

and we have lock again if n *is even (just with a* $n\pi$ *phase offset). However, if* n *is odd, then what pops out of the sine is MINUS* Δ *and that leads to INSTABILITY because the homogeneous equation will have a POSITIVE exponent. But that can't last long because the instability will make* $\phi(t)$ *grow so that we move up to the next value of* n*. So, any* n *odd gets kicked to the nearest* n *even solution. So the PLL is all about STABILITY!*

This system LOCKS ON to the input sinusoid BECAUSE it's a stable system (stable differential equation). The negative feedback stabilizes things just like negative feedback stabilizes op-amp circuits. Negative feedback is NEAT!

(b) Given the same input $\cos(2\pi f_c t)$ is it possible for the phase locked loop to "lock on" to frequencies other than f_c. If so, which ones? If not, why not?

HINT: Think about the bandwidth of the low pass filter – which we never actually specified in the previous part.

SOLUTION: *Let's start with the previous part, but not assume* $e(t)$ *is NOT small. The input to the low pass filter is*

$$\cos^2(2\pi f_c t)\sin e(t) + \cos(2\pi f_c t)\sin(2\pi f_c t)$$

Certainly we're gonna lose the really high frequency second term and the cosine squared term averages to $1/2$ *so*

$$\dot{\phi}(t) = \frac{1}{2}\sin e(t)$$

Now, the question "becomes how low is low on the low pass filter?"

Let's do some thought experiments. Suppose the PLL (that's technicalese for phase locked loop) frequency starts out too low ($f_0 \ll f_c$). Then $e(t)$ is large and $\dot{\phi} = 0$ because $\sin e(t)$ never makes it out of the LPF. $\dot{\phi} = 0$ implies zero frequency so the VCO output goes even lower until it's a D.C. level. No lock!

We can make the same argument if the VCO starts too high in frequency too. So, there's a RANGE of frequencies which can be locked in. Let's say the low pass filter has bandwidth W Hz (single sided). Then we know that $|\frac{de(t)}{dt}| < 2\pi W$ to achieve lock. This means that the initial VCO frequency f_0 has to $|f_0 - f_c| < W$ Otherwise the VCO outputs whatever it's lowest frequency is and thinks it's doing a good job!

If you're REALLY interested in PLLs, our local expert is Dave Daut. He can tell you as much as (or more than) you'll ever want to know about PLLs since he was heavy into the topic when it was hot quite some time ago – and still teaches the rudiments in his senior level course on RF engineering. The down side is that PLLs were studied to death quite a long time ago and are essentially a dead research topic. But so are vacuum tube audio amplifiers and they're still fun to learn about and use.