

Haykin: 4.1-4.4,5.1-5.7

1. Which of the following signals are orthogonal? You must show all work.

(a) $\cos t$ and $\sin t$ on $(0, \pi)$.

SOLUTION: Antisymmetric about $\pi/2$ so when you integrate product you get zero. **ORTHOGONAL**

(b) $\cos t$ and $\sin t$ on $(0, \pi/2)$.

SOLUTION: Product is $\frac{1}{2} \sin 4\pi t$. Only one positive hump occurs in $(0, \pi/2)$ so the integral is nonzero. **NOT ORTHOGONAL**

(c) t^2 and t^3 on $(-1, 1)$.

SOLUTION: Product has odd symmetry so integral on symmetric interval about zero will be zero. **ORTHOGONAL**

(d) $t \cos 2\pi t$ and $\cos 2\pi t$ on $(-\pi, \pi)$.

SOLUTION: Same idea (product has odd symmetry). **ORTHOGONAL**

(e) $te^{-|t|}$ and $t^2e^{-t^2}$ on $(-1, 1)$.

SOLUTION: Same idea again (product has odd symmetry). **ORTHOGONAL**

2. Consider the signal $s(t)$ shown in Figure P4.1 (page 300, Haykin).

(a) Determine the impulse response of a filter matched to this signal and sketch it as a function of time.

SOLUTION: The impulse response of the matched filter is

$$h(t) = s(T - t)$$

Both waveforms are plotted in FIGURE 1

(b) Plot the matched filter output as a function of time.

SOLUTION: Output of the matched filter is obtained by convolving $h(t)$ with $s(t)$. The result is shown in FIGURE 2

(c) What is the peak value of output.

SOLUTION: From FIGURE 2 it is clear that peak value of filter output is equal to $A^2T/4$ and occurs at $t = T$.

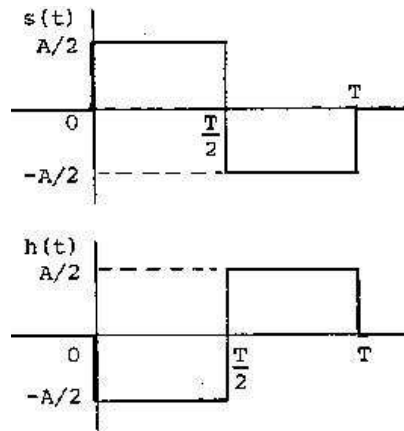


Figure 1: waveforms

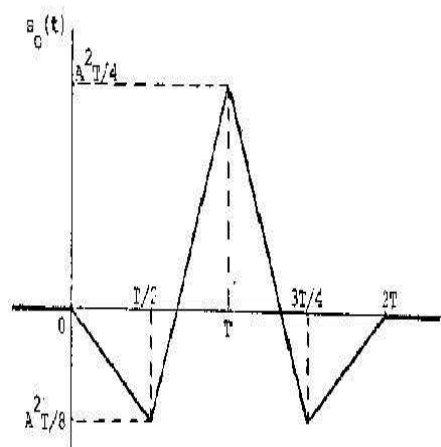


Figure 2: Matched filter output waveform

3. Figure P4.2a (page 301, Haykin) shows a pair of pulses that are orthogonal to each other over the interval $[0, T]$. In this problem we investigate the use of this pulse pair to study a two-dimensional matched filter.

(a) Determine the matched filters for pulses $s_1(t)$ and $s_2(t)$ considered individually.

SOLUTION: The matched filter $h_1(t)$ for pulse $s_1(t)$ is given in solution to previous problem. For $h_2(t)$, which is matched to $s_2(t)$,

$$h_2(t) = s_2(T - t)$$

whis is represented by FIGURE 3

(b) Form a two dimensional matched filter by connecting two of the matched filters of part 1 in parallel, as shown in figure P4.2b (page 301, Haykin). Hence, demonstrate the following:

i. When the pulse $s_1(t)$ is applied to the two-dimensional matched filter, the response of the lower matched filter (sampled at time T) is zero.

SOLUTION: The response of the matched filter, matched to $s_2(t)$ and due to $s_1(t)$

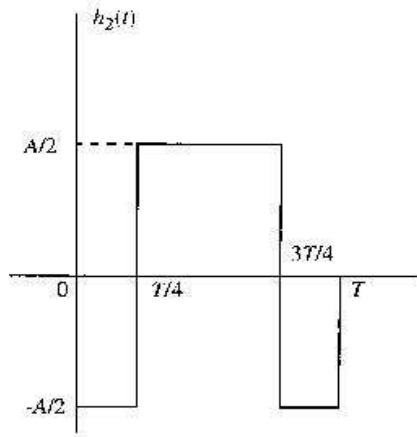


Figure 3: Matched filter output waveform

as input, is obtained by convolving $h_2(t)$ with $s_1(t)$, as shown by

$$y_{21}(t) = \int_0^T s_1(\tau)h_2(t - \tau)d\tau$$

The waveform $y_{21}(t)$ is shown in FIGURE 4. From the figure it is clear that $y_{21}(T) = 0$. This figure also includes the corresponding waveforms of input $s_1(t)$ and impulse response $h_2(t)$.

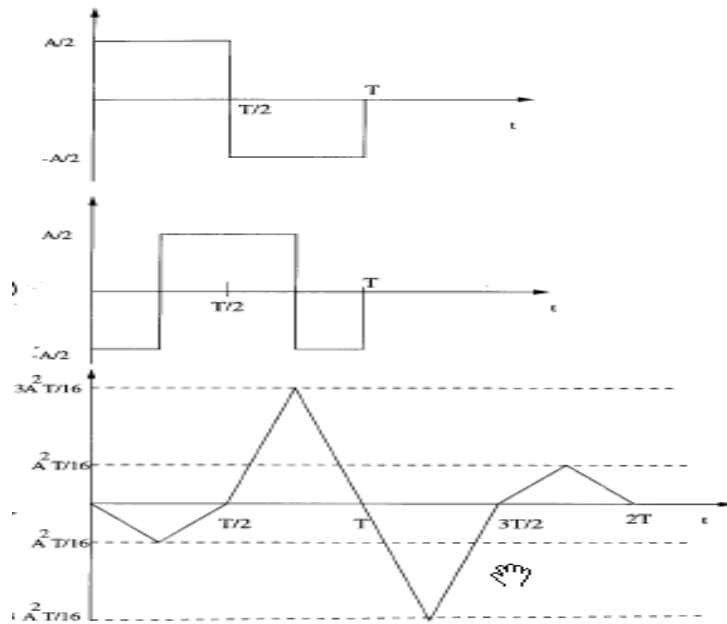


Fig. 3

Figure 4: Matched filter output waveform

- ii. When the pulse $s_2(t)$ is applied to the two-dimensional matched filter, the response of the upper matched filter (sampled at time T) is zero.

SOLUTION: Next, the response of the matched filter, matched to $s_1(t)$ and due to $s_2(t)$ as input, is obtained by convolving $h_1(t)$ with $s_2(t)$, as shown by

$$y_{12}(t) = \int_0^T s_2(\tau)h_1(t - \tau)d\tau$$

FIGURE 5 shows the corresponding waveforms.

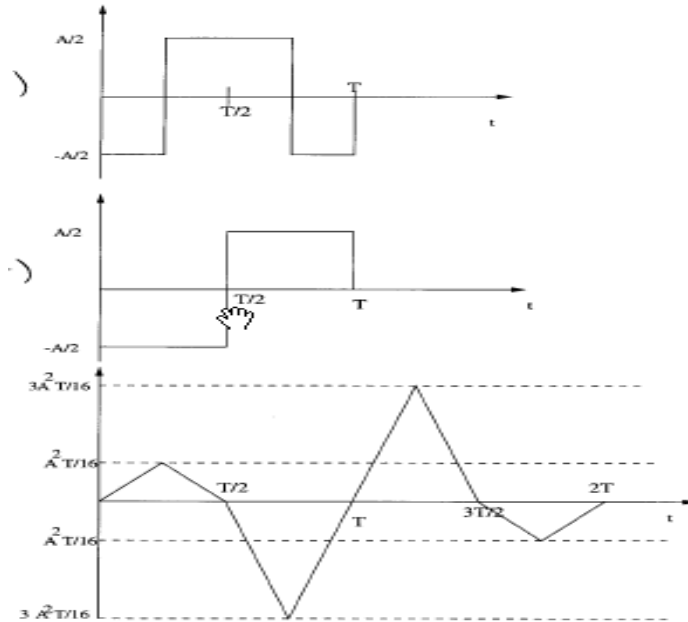


Figure 5: Matched filter output waveform

(c) How would you design a matched filter bank for a set of n orthogonal pulses.

SOLUTION: For n pulses that are orthogonal to each other over the interval $[0, T]$, the n -dimensional matched filter has the structure given by FIGURE 6, where the i th matched filter is matched to the i th waveform $s_i(t)$.

4. **Cora and the Tiger:** You are placed in a room with two doors and loudspeakers. You have been told that behind one door is untold wealth and behind the other, certain death. Of course, you are only told that any given door is equally likely to contain certain death or riches and that you MUST choose a door. Noise is played over the loudspeakers (really good speakers!) to confuse your thinking.

Luckily, your friend Cora the Communications Engineer has found a way to determine which door is which and together you've worked out a signaling system. If it's door 1 has riches, she'll whistle at some frequency ω_c . If door 2 has riches, she'll not whistle at all. Of course, Cora is outside the room, can only whistle with limited amplitude and the noise is loud.

Assume that you know the available signal is $A \cos(\omega_c t) + w(t)$ when Cora is whistling and just $w(t)$ otherwise. Assume that $w(t)$ is white, Gaussian, zero mean and has spectral height $N_0/2$.

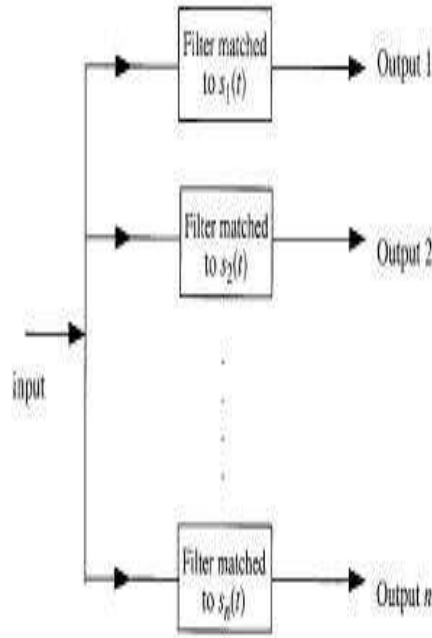


Figure 6: n-dimensional matched filter

- (a) If you can observe the signal for k cycles (on $[0, T = 2\pi k/\omega_c]$), what is the impulse response of the matched filter for the incoming signal $A \cos \omega_c t$? Please write your answer in analytic form.

SOLUTION: Let the received signal be $r(t)$ with

$$r(t) = \begin{cases} A \cos \omega_c t + w(t) & \text{Cora whistling} \\ w(t) & \text{Cora NOT whistling} \end{cases}$$

If the observation interval is $[0, T = 2\pi k/\omega_c]$, then the matched filter is a time reversed, scaled version of the information signal. In this case, owing to the symmetry of cosine, the time-reversed version is simply $h(t) = \cos \omega_c t$. Note that if k were not an integer then we'd have to be more careful and write $h(t) = \cos \omega_c (T - t)$.

- (b) What is the expected value of the matched filter output at time T given the signal is absent? Show your work.

SOLUTION: Matched filter output is the convolution of the input signal and $h(t)$. When no whistling we have

$$E[y(T)] = \int_0^{2\pi k/\omega_c} E[w(\tau)] \cos \omega_c (T - \tau) d\tau = 0$$

- (c) What is the expected value of the matched filter output at time T given the signal is present? Show your work.

SOLUTION: Likewise when Cora is whistling we have

$$\int_0^{2\pi k/\omega_c} A \cos \omega_c \tau \cos \omega_c (T - \tau) d\tau + \int_0^T E[w(\tau)] \cos \omega_c (T - \tau) d\tau = \int_0^T A \cos^2 \omega_c \tau d\tau = TA/2$$

- (d) What is the PDF of the matched filter output at time T given the signal is absent? Show your work.

SOLUTION: $y(T)$ will be a Gaussian random variable since passing a white Gaussian process through a linear filter gives gaussian samples at the output. We already know the mean is zero. The variance of $y(T)$ is simply $E[y^2(T)]$. So

$$E[y^2(T)] = \int_0^T \int_0^T E[w(\tau)w(\gamma)] \cos \omega_c \tau \cos \omega_c \gamma d\gamma d\tau$$

but since $w(t)$ is white we have $E[w(t)w(t + \tau)] = \frac{N_0}{2} \delta(\tau)$. Thus we can simplify the integral to

$$E[y^2(T)] = \int_0^T \int_0^T \frac{N_0}{2} \cos^2 \omega_c \tau d\tau = \frac{TN_0}{4}$$

Since we have the mean and variance AND know the variable is gaussian, we have

$$f_{Y(T)|no\ whistle}(y(T)|no\ whistle) = \frac{1}{\sqrt{\pi TN_0/2}} e^{-y^2(T)/(TN_0/2)}$$

- (e) What is the PDF of the matched filter output at time T given the signal is present? Show your work.

SOLUTION: Same idea except the mean of the gaussian is $AT/2$ (from part c).

$$f_{Y(T)|whistle}(y(T)|whistle) = \frac{1}{\sqrt{\pi TN_0/2}} e^{-(y(T)-AT/2)^2/(TN_0/2)}$$

- (f) If your ears can act as a matched filter for the whistle-signal and you wish to be at least 99.999999% sure that you make the right choice of door, how few cycles k can you wait before making a choice? You may use the fact that $\frac{1}{2} \text{erfc}(\sqrt{11}) \approx 10^{-6}$. Show your work. NOTE: $\text{erfc}(x) = 2 \int_x^\infty e^{-z^2} dz$.

SOLUTION: Equiprobable doors means a whistle/no whistle equally likely and we set our decision threshold at $AT/4$ (midway between $AT/2$ and 0). Probability of error is the area under the whistle-conditional distribution scaled by the probability of a whistle plus the area under the nowhistle-conditional distribution scaled by the probability of no whistle. Since whistle/no whistle equally likely and the areas under the tails are identical by symmetry we have

$$P_e = \int_{AT/4}^\infty \frac{1}{\sqrt{\pi TN_0/2}} e^{-z^2/(TN_0/2)} dz = \frac{1}{2} \text{erfc}\left(\sqrt{\frac{A^2 T^2 / 16}{TN_0/2}}\right) = \frac{1}{2} \text{erfc}\left(\sqrt{\frac{A^2 T}{8N_0}}\right)$$

So we need $\frac{A^2 T}{8N_0} = 11$. Substituting for k we have $\frac{A^2 2\pi k / \omega_c}{8N_0} = \frac{A^2 \pi k / \omega_c}{4N_0} = 11$ so that

$$k = \frac{44\omega_c N_0}{A^2 \pi}$$

- (g) Suppose Cora's a little drowsy and instead of whistling $\cos \omega_c t$ she whistles (unknownst to you) $\sin \omega_c t$. What is the probability that you choose the door with riches? Show your work.

SOLUTION: *If Cora whistles $A \sin \omega_c t$ then the output of the matched filter in response to the whistle will be zero whether Cora whistles or not; i.e.,*

$$E[y(T)] = \int_0^{2\pi k/\omega_c} A \sin \omega_c \tau \cos \omega_c (T - \tau) d\tau = 0$$

since $\sin \omega_c t$ orthogonal to $\cos \omega_c t$ on $(0, T)$ (k an integer). Since the whistle now gives absolutely no information about which door has the treasure, whichever door we choose has probability 1/2 of containing the treasure.

5. Cora the communications engineer is the operator of an early warning sonar system. The system takes sonar input $f(t)$ and produces a number, Γ as

$$\Gamma = \int_0^2 f(t)c(t)dt$$

where $c(t)$ is some fixed signal. Notice that Γ is simply the correlation $R_{fc}(0)$ on the interval $(0, 2)$ for real signals $f(t)$ and $c(t)$.

Incoming enemy subs produce the signal $e(t) = \sin \pi t$. Incoming allied subs produce $a(t) = \cos \pi t$. Thus, $f(t)$ can be the sum of each, both or neither signal; i.e.,

I no subs present $\rightarrow f(t) = 0$

II enemy sub only $\rightarrow f(t) = e(t)$

III allied sub only $\rightarrow f(t) = a(t)$

IV both subs present $\rightarrow f(t) = a(t) + e(t)$

- (a) Which of the following signals should be chosen for $c(t)$ if Cora wishes to have Γ nonzero ONLY when an enemy submarine is present? Assume that both an enemy sub and an allied sub might be present at the same time. You must justify your answer quantitatively.

- $\cos \pi t$
- $\cos(\pi t + \pi/4)$
- $\cos(\pi t - \pi/4)$
- $\sin 8\pi t$
- $u(t) - 2u(t - 1) + u(t - 2)$

SOLUTION: *Idea is that you want $c(t)$ to be orthogonal to allied sub signal but not orthogonal to enemy sub signal (all on $(0, 2)$). The signal $u(t) - 2u(t - 1) + u(t - 2)$ is orthogonal to $a(t)$ but not orthogonal to $e(t)$. So that's the one we choose. Just to be complete: $\cos \pi t$ can't work since it's the same as $a(t)$. Both $\cos(\pi t + \pi/4)$ and $\cos(\pi t - \pi/4)$ can be written as a sum of $\cos \pi t$ and $\sin \pi t$ and are therefore orthogonal to neither signal.*

- (b) Suppose Cora wants to be able to distinguish events I through IV based on the value of Γ : Which $c(t)$ should she use? You must justify your answer quantitatively.

SOLUTION: Now you want a different value for each different occurrence so you need something that is not orthogonal to either signal. Our candidates are therefore $\cos(\pi t + \pi/4)$ and $\cos(\pi t - \pi/4)$.

We see that

$$\cos(\pi t + \pi/4) = \frac{\cos \pi t}{\sqrt{2}} + \frac{\sin \pi t}{\sqrt{2}}$$

Likewise we see that

$$\cos(\pi t - \pi/4) = \frac{\cos \pi t}{\sqrt{2}} - \frac{\sin \pi t}{\sqrt{2}}$$

Let the inner product of $\frac{\sin \pi t}{\sqrt{2}}$ and $\sin \pi t$ be Q . Then the inner product of $\frac{\cos \pi t}{\sqrt{2}}$ and $\cos \pi t$ will be Q as well.

So we have a problem. We cannot distinguish between both subs being present or no subs present using $c(t) = \cos(\pi t - \pi/4)$ and we can't distinguish between EITHER enemy or ally with $c(t) = \cos(\pi t + \pi/4)$.

NONE OF THE $c(t)$ will work.