RUTGERS

School of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2011**

FINAL Examination

There are **FIVE** problems. Each problem subpart is stated on a different sheet. Show all work on the stapled sheets provided (front and back). DO NOT DETACH THE SHEETS. You are allowed THREE sides of $8.5 \times 11\text{in}^2$ paper handwritten note sheets.

1. *(30 points)* **Signal Space**

   Let $\phi_m(t) = u(t-m) - u(t-(m+1))$, $m = 0, 1, ..., M$

   (a) *(10 points)* Verify that the $\phi_m(t)$ are mutually orthonormal.

   SOLUTION: *The waveforms do not overlap in time, so their products will be zero. This means that they are orthogonal. As for normality, $\phi_m^2(t) = \phi_m(t)$ and each $\phi_m(t)$ has unit area, so the functions also have unit energy – orthonormal.*

   (b) *(10 points)* We represent functions $s(t)$ as points in the signal space described by the $\phi_m(t)$. For $M = 5$ please *carefully* sketch the waveform corresponding to the signal vector $\mathbf{s} = [1, 0, \frac{1}{2}, -2, 0, 0]$

   SOLUTION:

   (c) *(10 points)* Suppose again that $M = 5$ and

   $$s(t) = u(t) - 2u(t-2) + u(t-6)$$

   Please provide a signal vector $\mathbf{s}$ corresponding to the signal $s(t)$.

   SOLUTION:

   $$\mathbf{s} = [1, 1, -1, -1, -1, -1]$$

2. *(30 points)* **Multi-Hypothesis Testing**

   Suppose you have three events H_0, H_1 and H_2 and $\exists$ r.v. R which is related to these hypotheses by conditional densities, $f_{R|H_0}(r|H_0)$, $f_{R|H_1}(r|H_1)$ and $f_{R|H_2}(r|H_2)$. Furthermore, assume that these three events have probabilities p_0, p_1 and p_2. Your job is to observe R and then make a guess about which of the three events occurred. Your performance metric is probability of error.

   (a) *(10 points)* Please provide an expression for the probability of error, P_e, in terms of $\text{Prob}[\text{say}H_i|H_j]$, $i, j = 0, 1, 2$ and the event probabilities p_i, $i = 0, 1, 2$.

   SOLUTION:

   $$P_e = p_0\left(Prob[sayH_1|H_0] + Prob[sayH_2|H_0]\right) + p_1\left(Prob[sayH_0|H_1] + Prob[sayH_2|H_1]\right) + p_2\left(Prob[sa\right.$$

or

$$P_e = p_0\left(1 - \textit{Prob}[sayH_0|H_0]\right) + p_1\left(1 - \textit{Prob}[sayH_1|H_1]\right) + p_2\left(1 - \textit{Prob}[sayH_2|H_2]\right)$$

(b) *(10 points)* Suppose $p_2 = 0$. Please provide a decision rule based on the observed value r which minimizes the probability of error.

SOLUTION: *There are only two hypotheses so we need only consider cases* 0 *and* 1.

$$\frac{p_0 f_{R|H_0}(r|H_0)}{(1-p_0) f_{R|H_1}(r|H_1)} \overset{H_0}{\underset{H_1}{\gtrless}}$$

(c) *(10 points)* Now suppose $p_i \neq 0$, $i = 0, 1, 2$. Please *derive* decision rule***s*** based on the observed value r which minimize the probability of error. You must explain your answer quantitatively.

SOLUTION: *The same logic we applied for two regions applies for M regions. Consider a value of r. We increase the probability of error least if we assign r to the region in which expression $p_i\left(1 - \textit{Prob}[sayH_i|H_i]\right)$ is least which means that region in which $p_i \textit{Prob}[sayH_i|H_i]$ is largest.*

This reasoning leads to:

$$\frac{p_0 f_{R|H_0}(r|H_0)}{p_1 f_{R|H_1}(r|H_1)} \overset{H_0}{\underset{H_1}{\gtrless}}$$

$$\frac{p_0 f_{R|H_0}(r|H_0)}{p_2 f_{R|H_2}(r|H_2)} \overset{H_0}{\underset{H_2}{\gtrless}}$$

$$\frac{p_1 f_{R|H_1}(r|H_1)}{p_2 f_{R|H_2}(r|H_2)}$$

3. *(30 points)* **Binary Signaling**

A signal space is described by two orthonormal waveforms $\phi_1(t)$ and $\phi_2(t)$. A binary communication system is built using equiprobable transmitted signals $s_0(t) = 3\phi_1(t) + 4\phi_2(t)$, $s_1(t) = -s_0(t)$.

At the receivers we have

$$r(t) = s_i(t) + w(t)$$

where $w(t)$ is a zero mean white Gaussian noise processes with spectral height $N_0 = 1$.

(a) *(10 points)* Please provide a detailed sketch of the minimum probability of error receiver for this system. What is the signal to noise ratio for this system? Provide (or derive) an expression for the probability of error, P_e.

HINT: You might want to sketch the signal points in signal space.

SOLUTION: *You have a correlator receiver which beats the incoming $r(t)$ against the basis functions $\phi_i(t)$. The signal energy is* 25 *while the noise is* 2. *If we define the CCDF of the unit variance zero mean Gaussian as $\Psi(x)$ then $P_e = \Psi(\sqrt{25/2})$.*

(b) *(10 points)* Suppose the signal constellation is changed to $s_0(t) = 4\phi_1(t) + 3\phi_2(t)$, $s_1(t) = -s_0(t)$. What is the probability of error for this new system? You must justify your result.

SOLUTION: *In white gaussian noise, the probability of error depends only on the distance between the signals. Since the new signal constellation is simply a rotation of the old, the probability of error will be the same as in the previous part.*

(c) *(10 points)* Let $w_1 = \int w(t)\phi_1(t)dt$ and $w_2 = \int w(t)\phi_2(t)dt$. w_1 and w_2 are still independent Gaussian random variables, but while $\sigma^2_{w_1} = 1$, we now have $\sigma^2_{w_2} = 2$. That is, the noise is no longer white.

Is the probability of error using signal constellation $s_0(t) = 3\phi_1(t) + 4\phi_2(t)$, $s_1(t) = -s_0(t)$ the same as that for signal constellation $s_0(t) = 4\phi_1(t) + 3\phi_2(t)$, $s_1(t) = -s_0(t)$? Why/why not? You answer MUST be explicit.

SOLUTION: *The noise variances are NOT equal so we need to go back to the LRT. The easiest approach is to first change variables so that we have white noise again – this approach is called "whitening" (kinda obviously :)).*

$$r'_1 = r_1$$
$$r'_2 = r_2/\sqrt{2}$$

so that we have

$$r'_1 = s_{i1} + w_1$$

and

$$r'_2 = \frac{1}{\sqrt{2}} s_{i2} + w'_2$$

Since the w_i are independent, scaling by a constant leaves w_1 independent of w'_2 – and owing to the scaling, both noise terms have unit variance and both are Gaussian. So now, we can look at distance again. For $\mathbf{s}_0 = [3, 4]$ we have $\mathbf{s}'_0 = [3, 4/\sqrt{2}]$ whose energy is $9 + 8 = 17$. For $\mathbf{s}_0 = [4, 3]$ we have $\mathbf{s}'_0 = [4, 3/\sqrt{2}]$ whose energy is $16 + 4.5 = 20.5$ – which has larger energy and thus has lower signal to noise.

4. *(30 points)* **Orthonormal Basis Sets are EVERYWHERE!**

You are given a signal $m(t)$ which is band limited to $\pm B$ Hertz. We take samples $m_k = m(k\Delta)$ and we know that we can reconstruct the signal $m(t)$ perfectly by forming

$$z(t) = \sum_k m_k \alpha \delta(t - k\Delta)$$

(where Δ is the Nyquist sampling interval and α a constant) and passing $z(t)$ through a low pass filter with cutoff frequency B Hertz. That is, if $h(t)$ is the ideal low pass filter response with unit height in frequency domain, then

$$m(t) = \sum_k m_k \alpha h(t - k\Delta)$$

(a) *(5 points)* What is the maximum value of Δ that will allow $m(t)$ to be reconstructed perfectly from the samples $\{m_k\}$?

SOLUTION: *Nyquist* – $\Delta < 1/2B$

(b) *(5 points)* What value of α makes $\alpha h(t - k\Delta)$ have unit energy?

HINT:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

where $X(f)$ is the Fourier Transform of $x(t)$.

SOLUTION: *The energy in $h(t)$ is $2B\alpha^2$. So, we must have $\alpha = \frac{1}{\sqrt{2B}}$. The energy will be the same no matter where we shift $h(t)$.*

(c) *(10 points)* Please show that if the $\alpha h(t - k\Delta)$ have unit energy, then they are also mutually orthonormal.

HINT:

$$\int_{-\infty}^{\infty} s(t)g(t)dt = \int_{-\infty}^{\infty} S(f)G^*(f)df$$

where $S(f)$ and $G(f)$ are the Fourier transforms of real signals $s(t)$ and $g(t)$ respectively. I shoulda made you derive this, but I took mercy.

SOLUTION: *If you try this in time domain, once again, you're nuts.*

$$H_k(f) = e^{-j2\pi f k\Delta} H_0(f)$$

So,

$$H_k(f)H_\ell^*(f) = e^{-j2\pi f(k-\ell)\Delta}|H_0(f)|^2 = e^{-j2\pi \frac{f}{2B}(k-\ell)}$$

which exists only on the interval $f \in (-B, B)$. We have a sinusoidal function of f integrated over an integral number of periods if $k \neq \ell$ so the integral is zero for $k \neq \ell$. So the $\alpha h(t - k\Delta)$ are orthogonal. Since $\alpha h(t - k\Delta)$ has unit energy, the ensemble of signals is orthonormal.

(d) *(10 points)* For your value of α please show that

$$m(t) = \sum_k m_k \alpha h(t - k\Delta)$$

and then derive an expression for the m_k *solely* in terms of $m(t)$ and $\alpha h(t - k\Delta)$.

(The correct result should freak you out a little.)

(No credit for freaking out though. :))

SOLUTION: *The ensemble $\{\alpha h(t - k\Delta)\}$ are orthonormal. So, we can obtain the projections*

$$m_k = \int_{-\infty}^{\infty} \alpha m(t) h^*(t - k\Delta)dt$$

from which the reconstruction

$$m(t) = \sum_k m_k \alpha h(t - k\Delta)$$

follows.

We show equivalence of $m_k = m(k\Delta)$ by noting that

$$m_k = \int_{-\infty}^{\infty} \alpha m(t)h^*(t-k\Delta)dt = \int_{-\infty}^{\infty} \alpha M(f)H^*(f)e^{j2\pi f k\Delta}df = \alpha \int_{-B}^{B} M(f)e^{j2\pi f(k\Delta)}df$$

which is an inverse Fourier Transform with time argument $k\Delta$.

This is freaky because it says the projection of $m(t)$ onto the shifted sinc function (which is our low pass filter in time domain) is the same as sampling the signal at the peak of the sinc ($t = k\Delta$).

5. *(30 points)* **Cora Rolls With It**

Cora the communication engineer wishes to build a receiver for a QPSK system which decodes equiprobable signal points $s_0(t) = \sqrt{2/T}\cos 2\pi f_c t$, $s_1(t) = \sqrt{2/T}\sin 2\pi f_c t$, $s_2(t) = -\sqrt{2/T}\cos 2\pi f_c t$ and $s_3(t) = -\sqrt{2/T}\sin 2\pi f_c t$ where the symbol interval is $T = 1/f_c$. Each signal signifies a pair of bits corresponding to the signal index. So, $s_0(t) \rightarrow 00$, $s_1(t) \rightarrow 01$, $s_2(t) \rightarrow 10$, and $s_3(t) \rightarrow 11$.

However like you, she has completely forgotten how to make phase locked loops and cannot exactly recover the carrier signal at the receiver. Luckily she can generate $\cos(2\pi f_c + \theta)$ and $\sin(2\pi f_c + \theta)$. Unfortunately, she never knows the value of θ is ahead of time and in addition, θ drifts slowly (as compared to a symbol interval) over time.

You will help Cora design a system that can decode information even in the presence of phase errors.

(a) *(10 points)* Plot out the transmitted signal constellation in a suitable signal space and label the points with the corresponding pair of bits. There is no noise in the system. Assume $\theta = 0$. What is the probability of error at the receiver?

SOLUTION: *Probability of error is zero since the point that's sent is the same one decoded*

(b) *(10 points)* Now assume that $\theta = \pi$?. Repeat the previous part.

SOLUTION: *Probability of error is one since the point that's sent is the exact opposite of the one decoded*

(c) *(10 points)* Now suppose that Cora's receiver can be adjusted to output phase *change* between successive symbol intervals. Please provide a signaling scheme that transmits $s_0(t)$ on the first symbol interval and then allows two bits to be correctly decoded after each subsequent transmission by examining the phase difference between successive transmissions. Again, assume there is no noise in the system.

SOLUTION: *The signals are $+\pi/2$ (transmit the signal that is one position clockwise from the previous), $-\pi/2$ (counterclockwise one position), $\pm\pi$ (skip a position for the next interval – $-\pi$ and π are indistinguishable) and 0 (transmit the same signal as the previous interval). Four signals sent equiprobably, two bits.*