School of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2009**

FINAL Examination

There are **FOUR** problems. Each problem subpart is stated on a different sheet. Show all work on the stapled sheets provided (front and back). DO NOT DETACH THE SHEETS. You are allowed THREE sides of $8.5 \times 11\text{in}^2$ paper handwritten note sheets.

1. **Cora the Squirliminator:** Marty, still mad at Cora for her plot to detonate squirrel-nukes in the pit, seeks revenge by throwing stones at one side of Cora's home. However, Cora uncovers Marty's plot and determines that Marty will send out one of FOUR equiprobable messages $s_i(t)$ $i = 1, 2, 3, 4$ to inform his buddies that they should attack from North, South, East, or West. The message signals are defined as:

$$s_i(t) = \sqrt{2E}\cos\left(2\pi t + (2i-1)\frac{\pi}{4}\right), \quad 0 \le t \le 1 \quad \text{where i} = 1, 2, 3, 4;$$

corresponding to North, South, East and West respectively. The messages are all of 1 second duration.

Your job is to help Cora properly decode the attack signals so that she has time to don her very heavy **Squirliminator**(tm) suit and appear at the appropriate window.

(a) *(10 points)* Let
$$\phi_1(t) = \sqrt{2}\cos(2\pi t), \quad 0 \le t \le 1$$
and
$$\phi_2(t) = \sqrt{2}\sin(2\pi t), \quad 0 \le t \le 1$$
Prove that $\phi_1(t)$ and $\phi_2(t)$ are orthonormal functions, i.e.;
$$\int_0^1 \phi_i(t)\phi_j(t)dt = \begin{cases} 1 & \text{if i = j} \\ 0 & \text{otherwise} \end{cases}$$

(b) *(10 points)* Are all the $s_i(t)$ mutually orthogonal? Are some of them mutually orthogonal? Why, why not?

(c) *(20 points)* Assume the signals are corrupted by zero mean white Gaussian noise with spectral height N_0. Please design an optimal reciever which will allow Cora to decode the direction of attack with minimum probability of error. Please provide a labeled block diagram of your receiver.

2. **Cora, Marty and DaMar:** Cora the communications engineer has a dog named Data. As you know by now, Cora and Marty are not exactly friendly. So it was with great dismay that Cora watched Data gave birth to DaMar, the squirrel-dog.

   DaMar is much more squirrel than dog and roams the nearby forest foraging for nuts just like Marty. Cora, seeking revenge, wants to drive Marty from the forest and she's decided to destroy Marty's nut caches. However, she has an auntly soft spot for DaMar and does not want to destroy his caches.

   Now, DaMar is a rather large squirrel and collects more nuts on average than Marty does. In fact the number of nuts N in a given cache is

   $$f_N(n|\text{DaMar}) = \frac{(\lambda_D)^n}{n!}e^{-\lambda_D}$$

   for DaMar and

   $$f_N(n|\text{Marty}) = \frac{(\lambda_M)^n}{n!}e^{-\lambda_M}$$

   for Marty where $\lambda_D = \alpha\lambda_M$ and $\alpha > 1$

   When Cora comes across a nut cache, she counts the number of nuts N and makes a decision about whether to destroy the cache. Your job it to help her devise an appropriate decision rule. You may assume that Cora comes across a cache belonging to Marty or one belonging to DaMar with equal probability.

   (a) *(10 points)* Based on n the number of nuts in the cache, please provide a decision rule which minimizes the probability that Cora makes a mistake. A mistake is defined as when Cora destroys a cache belonging to DaMar or leaves a cache belonging to Marty.

   (b) *(20 points)* For $\alpha = e^2$ and $\lambda_M = 1$, please provide an analytic expression for the probability that Cora makes a mistake using your decision rule. (NOTE: $e^2 \approx 7.39$.) What happens to this probability if $\alpha = e^4$. (NOTE: $e^4 \approx 54.6$.)

   (c) *(10 points)* Suppose Cora is only worried about the probability that she destroys one of DaMar's caches and does not worry at all about erroneously leaving one of Marty's. What is her optimal policy in that case? What is the probability she mistakenly destroys one of DaMar's caches?

3. **Orthonormal Basis Sets are EVERYWHERE!** You are given a signal $m(t)$ which is band limited to $\pm B$ Hertz. We take samples $m_k = m(k\Delta)$ and we know that we can reconstruct the signal $m(t)$ perfectly by forming

   $$z(t) = \sum_k m_k \alpha \delta(t - k\Delta)$$

   (where Δ is the Nyquist sampling interval and α a constant) and passing $z(t)$ through a low pass filter with cutoff frequency B Hertz. That is, if $h(t)$ is the low pass filter response with unit height in frequency domain, then

   $$m(t) = \sum_k m_k \alpha h(t - k\Delta)$$

(a) *(10 points)* What is the maximum value of Δ that will allow $m(t)$ to be reconstructed perfectly from the samples $\{m_k\}$?

(b) *(10 points)* What value of α makes the $\alpha h(t-k\Delta)$ have unit energy?

HINT: You're a fool if you try to work in time-domain, and none of you are fools.

(c) *(10 points)* Please show that if the $\alpha h(t-k\Delta)$ have unit energy, then they are also mutually orthonormal.

HINT:

$$\int_{-\infty}^{\infty} s(t)g(t)dt = \int_{-\infty}^{\infty} S(f)G^*(f)df$$

where $S(f)$ and $G(f)$ are the Fourier transforms of real signals $s(t)$ and $g(t)$ respectively. I shoulda made you derive this, but I took mercy.

(d) *(10 points)* For your value of α please determine whether

$$m(t) = \sum_k m_k \alpha h(t-k\Delta)$$

and comment on the result.

4. **Noise Through a Fun House Mirror**

Suppose you can use two equiprobable signals (corresponding to vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ in some signal space) to communicate a single bit. With a correlation receiver you will receive $r_i = s_{ki} + w_i$, $i = 1, 2$ where s_{ki} is the i^{th} component of signal vector $\mathbf{s}_k$. Were the noise white, then the w_i would be independent and you would use distance-based decoding for the optimal minimum probability of error receiver.

However, suppose you are told that w_1 and w_2 are jointly gaussian with probability density

$$\left(\frac{1}{2\pi}\right)^{\frac{N}{2}} |\mathbf{K}|^{-\frac{1}{2}} e^{-\frac{1}{2}\mathbf{w}^\top \mathbf{K}^{-1}\mathbf{w}}$$

where $N = 2$ and

$$\mathbf{K} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{bmatrix}$$

and ρ is a constant between -1 and 1. Note that $|\mathbf{K}| = \sigma_1^2\sigma_2^2(1-\rho^2)$ and

$$\mathbf{K}^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_1\sigma_2\rho & \sigma_1^2 \end{bmatrix}$$

(a) *(20 points)* Given equiprobable signal vectors $\mathbf{s}_1$ and $\mathbf{s}_2$, please derive an ANALYTIC decision rule for deciding which was sent in terms of the relevant conditional probability densities on the received signal vector $\mathbf{r}$, the covariance matrix $\mathbf{K}$ and the signal vectors $\mathbf{s}_1$ and $\mathbf{s}_2$.

(b) *(10 points)* Suppose $\rho = 0$ and $\sigma_1^2 = 2\sigma_2^2$. For $\mathbf{s}_1 = (1, 0)$ and $\mathbf{s}_2 = (0, 1)$ please sketch the minimum probability of error decision region(s).

(c) *(10 points)* Suppose $\rho = \frac{1}{\sqrt{2}}$ and $\sigma_1^2 = \sigma_2^2$. For $\mathbf{s}_1 = (1, 0)$ and $\mathbf{s}_2 = (0, 1)$ please sketch the minimum probability of error decision region(s).

(d) *(10 points)* Suppose $\rho = \frac{1}{\sqrt{2}}$ and $\sigma_1^2 = 2\sigma_2^2$. For $\mathbf{s}_1 = (1, 0)$ and $\mathbf{s}_2 = (0, 1)$ please sketch the minimum probability of error decision region(s).

(e) *(10 points)* Suppose $\rho = 0$ and $\sigma_1^2 = 2\sigma_2^2$. You are now allowed to choose your signals $\mathbf{s}_1$ and $\mathbf{s}_2$ so as to minimize probability of error. However, energy is limited and you are constrained to have $|\mathbf{s}_k|^2 \leq \mathcal{E}$. Please choose signals that minimize the probability of error, plot them in the signal space and draw the appropriate decision region.