THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2005**

Final Exam

There are FIVE questions. You have the three hours to answer them. Read the WHOLE EXAM before doing the problems. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(50 points)* **Continuous Modulation:** A signal $r(t) = b(t)\cos 2000\pi t$ is received where

$$b(t) = \sum_{k=-\infty}^{\infty} b_k p(t - kT)$$

with $p(t) = \frac{\sin \pi t}{\pi t}$ and the b_k having arbitrary real values.

(a) *(25 points)* Can $b(t)$ be recovered exactly using a synchronous AM demodulator? If so, why and how? If not, why not?

SOLUTION: *$p(t)$ is a sinc pulse. It's Fourier transform has constant amplitude between ± 1Hz. $b(t)$ is a linear superposition of shifted versions of $p(t)$ and hence has a spectrum which exists only for $f \in (-1,1)$ as well. Thus, the signal $r(t)$ is a bandpass signal, centered around the carrier $f_c = 1000$Hz. Synchronous AM demodulation will therefore exactly recover the band-limited information waveform $b(t)$.*

(b) *(25 points)* Suppose now that $p(t) = u(t) - u(t-1)$. Can $b(t)$ be recovered exactly using a synchronous AM demodulator? If so, why and how? If not, why not?

SOLUTION: *$P(f)$ is a sinc spectrum which exists for all frequencies. So, the spectrum of $r(t)$ is sinc convolved with $\frac{1}{2}(\delta(f+1000) + \delta(f-1000))$ and synchronous demodulation will produce a copy of the information spectrum $P(f)$, but there will also be copies at $P(f-2000)$ and $P(f+2000)$ whose tails will overlap with $P(f)$. So $b(t)$ cannot be exactly recovered using synchronous AM demodulation.*

2. *(50 points)* **Cora, Marty and DaMar:** Cora the communications engineer has a dog named Data. As you know by now, Cora and Marty are not exactly friendly. So it was with great dismay that Cora watched Data gave birth to DaMar, the squirrel-dog.

DaMar is much more squirrel than dog and roams the nearby forest foraging for nuts just like Marty. Cora, seeking revenge, wants to drive Marty from the forest and she's decided to destroy Marty's nut caches. However, she has an auntly soft spot for DaMar and does not want to destroy his caches.

Now, DaMar is a rather large squirrel and collects more nuts on average than Marty does. In fact the number of nuts N in a given cache is

$$f_N(n|\text{DaMar}) = \frac{(\lambda_D)^n}{n!} e^{-\lambda_D}$$

for DaMar and

$$f_N(n|\text{Marty}) = \frac{(\lambda_M)^n}{n!}e^{-\lambda_M}$$

for Marty where $\lambda_D = \alpha\lambda_M$ and $\alpha > 1$

When Cora comes across a nut cache, she counts the number of nuts N and makes a decision about whether to destroy the cache. Your job it to help her devise an appropriate decision rule. You may assume that Cora comes across a cache belonging to Marty or one belonging to DaMar with equal probability.

(a) *(20 points)* Based on n the number of nuts in the cache, please provide a decision rule which minimizes the probability that Cora makes a mistake. A mistake is defined as when Cora destroys a cache belonging to DaMar or leaves a cache belonging to Marty.

SOLUTION: *This is a minimum probability of error problem so we form the likelihood ratio:*

$$\frac{\frac{(\lambda_M)^n}{n!}e^{-\lambda_M}}{\frac{(\lambda_D)^n}{n!}e^{-\lambda_D}} = \left(\frac{1}{\alpha}\right)^n e^{(\alpha-1)\lambda_M} \quad \mathop{\gtrless}_{\textit{save}}^{\textit{destroy}} \quad 1$$

Taking the log of both sides yields

$$n \quad \mathop{\gtrless}_{\textit{destroy}}^{\textit{save}} \quad \frac{1}{\log(\alpha)}(\alpha-1)\lambda_M$$

(Notice that the inquality changes when you multply through to remove the minus signs.)

(b) *(20 points)* For $\alpha = e^2$ and $\lambda_M = 1$, please provide an analytic expression for the probability that Cora makes a mistake using your decision rule. (NOTE: $e^2 \approx 7.39$.) What happens to this probability if $\alpha = e^4$. (NOTE: $e^4 \approx 54.6$.)

SOLUTION: *We have for* $\alpha = e^2$

$$n \quad \mathop{\gtrless}_{\textit{destroy}}^{\textit{save}} \quad \frac{1}{2}(e^2 - 1) = 3.2$$

and for $\alpha = e^4$

$$n \quad \mathop{\gtrless}_{\textit{destroy}}^{\textit{save}} \quad \frac{1}{4}(e^2 - 1) = 13.4$$

Cora makes an error when she destroys one of DaMar's caches and when she does not destroy one of Martin's. So

$$P_e = \frac{1}{2}\sum_{k=0}^{n^*} f_N(n|DaMar) + \frac{1}{2}\sum_{n^*+1}^{\infty} f_N(n|Marty)$$

where n^* *is the threshold value to the right side of the inequality.*

(c) *(10 points)* Suppose Cora is only worried about the probability that she destroys one of DaMar's caches and does not worry at all about erroneously leaving one of Marty's. What is her optimal policy in that case? What is the probability she mistakenly destroys one of DaMar's caches?

SOLUTION: *If she's not worried about not destroying one of Marty's caches but never wants to destroy one of DaMar's, Cora should simply not destroy any caches. Her probability of destroying one of DaMar's caches is identically zero.*

3. *(50 points)* **Digital Modulation:**

An information signal $b(t)$ has the form

$$b(t) = \sum_{k=-\infty}^{\infty} b_k p(t - kT)$$

where $p(t) = u(t) - u(t-T)$ (i.e., $p(t)$ is a unit-height pulse on $(0,T]$), the $b_k \in \{0,1\}$ are equiprobable random information digits and $T > 0$. Assume that the receiver knows the structure of the signal $b(t)$ but not the specific $\{b_k\}$.

(a) *(15 points)* If the received signal $r(t) = b(t)\cos\frac{2\pi}{T}t$, can you devise a receiver which recovers $b(t)$ EXACTLY if you have a copy of $\cos\frac{2\pi}{T}t$ available at the receiver. If so, how? If not, why not?

HINT: Try sketching an example of $r(t)$.

SOLUTION: *Incredibly easy: the local copy of* $\cos\frac{2\pi}{T}t$ *gives you a clock so you know the symbol intervals. And the symbols are* 0 *and* 1*. So, when the signal is non-zero in an interval,* $b_k = 1$*. If it's zero,* $b_k = 0$ *and this allows you to reconstruct* $b(t)$*.*

(b) *(15 points)*

Now suppose you only have some sinusoid $\cos(\frac{2\pi}{T}t + \phi)$ available at the receiver where ϕ is some phase offset. Can you still always recover $b(t)$ exactly from $r(t) = b(t)\cos\frac{2\pi}{T}t$? If so, how? If not, why not?

SOLUTION: *Just as incredibly easy: you don't have a time reference, but you still can see when* $r(t)$ *is identically zero and when it's not. Thus, you still immediately know what* $b(t)$ *is.*

(c) *(20 points)* Now suppose zero mean white Gaussian noise $n(t)$ with spectral height N_0 corrupts the received signal so that $r(t) = b(t)\cos\frac{2\pi}{T}t + n(t)$. Please carefully sketch and label a receiver which recovers the $\{b_k\}$ with minimum probability of error. You must also include the rule used for the "decision box". State all assumptions.

SOLUTION: *This is standard digital modulation. The signal*

$$s(t) = (u(t) - u(t-T))\cos\frac{2\pi}{T}t$$

and each interval $(kT, (k+1)T]$ *is a symbol interval. So we simply use a correlator receiver on each interval using* $s(t)$ *as the correlation signal. You could also use a matched filter since it's exactly equivalent to the correlator. As we've analyzed in class, the output of the correlator will be a Gaussian random variable. It's mean will be zero* *given $b = 0$ and $T/2$ given $b = 1$. In both cases its variance is the same. So, if X is the output of the correlator, the decision rule will be*

$$X \underset{b=0}{\overset{b=1}{\gtrless}} \frac{T}{4}$$

4. *(50 points)* **Quantization:** You are given the 2-bit quantizer shown in FIGURE 1 and are told that it's an optimal quantizer for a signal which has uniform probability density between zero and one. Is the quantizer optimal?

   (a) *(10 points)* Why is $Q(x)$ called a 2-bit quantizer?

   SOLUTION: *Because it has 4 levels and $2^b = 4$ when $b = 2$, or "two bits".*

   (b) *(40 points)* Is the quantizer optimal? If so, why? If not why not – and what IS the optimal quantizer?

   SOLUTION: *Loyd-Max: $x_k = (q_k + q_{k+1})/2$. For $k = 1$ we have $0.25 \neq (0.25 + 0.5)/2$ so we've got a problem – this quantizer is not optimal! However, if we just shift everything down by 0.125 (i.e., define a new quantizer $Q'(x) = Q(x) - 0.125$) we have*

$$\begin{array}{ll} k=1 & 0.25 = (0.125+0.375)/2 \\ k=2 & 0.5 = (0.375+0.625)/2 \\ k=3 & 0.75 = (0.625+0.875)/2 \end{array}$$

   and then we have to check the other part of Loyd-Max: $q_k = E[X|x \in \text{bin } k]$

$$\begin{array}{ll} k=1 & 0.125 = E[X|X \in (0,0.25)] \\ k=2 & 0.375 = E[X|X \in (0.25,05)] \\ k=3 & 0.625 = E[X|X \in (0.5,0.75)] \\ k=4 & 0.875 = E[X|X \in (0.75,1.0)] \end{array}$$

   and all is now well.

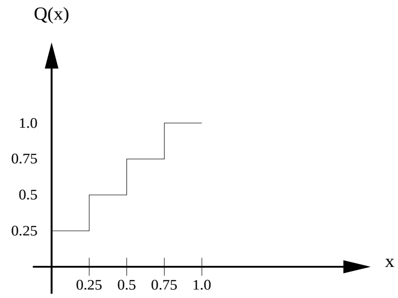

Figure 1: Quantizer for Problem 4

5. *(50 points)* **Signal Space:** You are given two signals on the interval $(0,1)$: $\phi_1(t) = u(t) - u(t-1)$ and $\phi_2(t) = u(t) - 2u(t-0.5) + u(t-1)$.

   (a) *(20 points)* Show that the $\phi_i(t)$ are orthonormal.

   SOLUTION: $\phi_1(t)\phi_2(t) = \phi_2(t)$ *and* $\int_0^1 \phi_2(t)dt = 0$ *so signals are orthogonal.* $\phi_i^2(t) = u(t) - u(t-1)$ *and* $\int_0^1 1dt = 1$ *so signals are also normal.*

   (b) *(30 points)* Provide an analytic expression for each of the signal points depicted in FIGURE 2. You may also sketch them if you like.

   SOLUTION: *Clockwise starting from upper right:* $s_1(t) = \phi_1(t) + \phi_2(t) = 2u(t) - 2u(t-0.5)$ *which is just* $\phi_2(t)$ *shifted up by* 1.

   $s_2(t) = \phi_1(t) + 0.5\phi_2(t) = 1.5u(t) - u(t-0.5) - 0.5u(t-1)$.

   $s_3(t) = 0.5(\phi_1(t) + \phi_2(t))$ *which is just* $s_1(t)/2$.

   $s_4(t) = 0.5\phi_1(t) + \phi_2(t) = 1.5u(t) - 2u(t-0.5) + 0.5u(t-1)$ *which is just* $\phi_2(t)$ *shifted up by* $1/2$

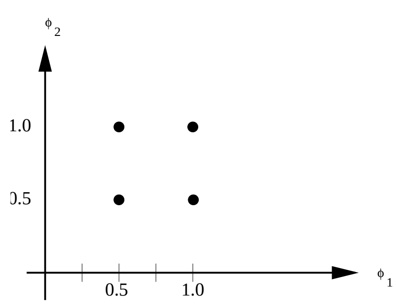

Figure 2: Signal points for Problem 5