THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:322 **Principles of Communications Systems** **Spring 2003**

Final Examination

There are four questions. You have three hours to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(35 points)* **Quick AM Fun:** Consider an AM system with program material $m(t)$ which has bandwidth (double-sided) $2B$. The receiver sees $r(t) = m(t)\cos(2\pi f_c t)$ and multiplies it by $f(t)$ followed by an ideal low pass filter with double-sided bandwidth $2B$ and gain 2 to obtain $y(t)$. What is $y(t)$ for

   (a) *(5 points)* $f(t) = \cos(2\pi f_c t)$, $B < f_c$
   SOLUTION: *Standard AM demodulation:* $y(t) = m(t)$

   (b) *(5 points)* $f(t) = \sin(2\pi f_c t)$, $B < f_c$
   SOLUTION: *Phase mismatch between modulating and demodulating sinusoids:* $y(t) = 0$.

   (c) *(5 points)* $f(t) = \cos(2\pi f_c t)$, $f_c < B < 2f_c$. You should sketch your result in lieu of an analytic expression.

   SOLUTION: *Modulation leads to overlap of spectra at the band edges. Demodulation does not get rid of it so you end up with an aliased baseband signal.*

   (d) *(5 points)* $f(t) = \sin(2\pi f_c t)$, $f_c < B < 2f_c$. You should sketch your result in lieu of an analytic expression.

   SOLUTION: *You end up with the band tips only.*

   (e) *(15 points)* Suppose for the case where $f_c < B < 2f_c$ you have available $f_1(t) = \cos(2\pi f_c t)$, $f_2(t) = \sin(2\pi f_c t)$ and a phase shifter (Hilbert filter $H(f) = j\,\text{sgn}(f)$) at the receiver. Develop a receiver structure which recovers $m(t)$ or show why $m(t)$ cannot be recovered. You may frame your answer with carefully labeled sketches.

   SOLUTION: *In part c) you got the baseband spectrum plus unwanted spectrum tips. In part d you got only the tips – one pointing up and the other pointing down, both multiplied by* j*. So you take the output of the receiver in d), pass it through a Hilbert filter and then add it to the output of the receiver in part c) to subtract off the unwanted band tips.*

2. *(40 points)* **Phase Locked Loops:** Consider a phase locked loop whose sole purpose is to lock on to the incoming sinusoid $\cos(2\pi f_c t)$ and produce $\sin(2\pi f_c t)$ at the output of the VCO (voltage controlled oscillator). The incoming sinusoid is multiplied by the output of the VCO and the result is sent through a low pass filter whose output is multiplied by $-K$ ($K > 0$ a constant). The output of the multiplier, call it $\dot{\phi}(t)$, is in turn sent to the input of the VCO which produces $\sin\phi(t)$.

(a) *(20 points)* Sketch a system diagram of the PLL and show it will produce the desired result if the gain K is large enough. You may assume that $\phi(t) \approx 2\pi f_c t$ and then make appropriate approximations.

SOLUTION: *The inputs to the multiplier are* $\cos(2\pi f_c t)$ *and* $\sin\phi(t)$. *We rewrite* $\phi(t) = 2\pi f_c t + e(t)$ *where* $e(t)$ *is assumed small. At the output of the multiplier we then have (after expanding the argument of the sine* $\cos^2(2\pi f_c t)\sin e(t) + \cos(2\pi f_c t)\sin(2\pi f_c t)$ *and the second term is lost when it passes through the low pass filter since it's HIGH frequency (*$\sin(4\pi f_c t)/2$ *to be exact). The first term has a D.C. component of* $1/2$ *(because of the cosine squared) and what pops out of the filter is* $\approx e(t)/2$ *since* $\sin x \approx x$ *for small* x.

We multiply this by $-K/2$ *and then have*

$$\dot{\phi}(t) = -Ke(t)/2 = -K(\phi(t) - 2\pi f_c t)/2$$

which we rewrite as

$$\dot{\phi}(t) + (K/2)\phi(t) = K\pi f_c t$$

This is a nice happy and stable first order differential equation whose homogeneous portion settles as $e^{-\frac{K}{2}t}$ *which for* K *large is FAST FAST FAST! So given the homogeneous solution dies out, we're left with the particular for* K *large of*

$$\phi(t) \approx 2\pi f_c t$$

which is exactly what we wanted in the first place! If yuo made it to here, you solved the problem and were done. The following is extra.

Now, above we assumed $e(t)$ *was small, but what if it's not small?!?! Specifically, what if* $e(t) = n\pi + \Delta(t)$ *where* $\Delta(t)$ *IS small but* n *can be ANY INTEGER!!!. All the same approximations hold, except that what pops out of the LPF is* $\pm\Delta(t)$, *not* $e(t)$. *We then have*

$$\dot{\phi}(t) = \pm K\Delta(t)/2 = \pm K(\phi(t) - 2\pi f_c t - n2\pi)/2$$

and we have lock again if n *is even (just with a* $n\pi$ *phase offset). However, if* n *is odd, then what pops out of the sine is MINUS* Δ *and that leads to INSTABILITY because the homogeneous equation will have a POSITIVE exponent. But that can't last long because the instability will make* $\phi(t)$ *grow so that we move up to the next value of* n. *So, any* n *odd gets kicked to the nearest* n *even solution. So the PLL is all about STABILITY!*

This system LOCKS ON to the input sinusoid BECAUSE it's a stable system (stable differential equation). The negative feedback stabilizes things just like negative feedback stabilizes op-amp circuits. Negative feedback is NEAT!

(b) *(20 points)* Is it possible for the phase locked loop to "lock on" to frequencies other than f_c. If so, which ones? If not, why not?

SOLUTION: *This was a poorly worded problem. What I WANTED to ask is whether with the same INPUT it could lock onto different frequencies (and the answer we've derived is NO from the analysis above). However, certainly we never specified* f_c *so if* f_c *is changed, CERTAINLY the loop can lock onto the new value. Thus, owing to professorial bumbling, either answer gets full credit.*

3. *(40 points)* **Quantization:** A program signal $x(t)$ at any time t has probability distribution $f_X(x)$. You are to design an optimal three level quantizer for this signal where the optimization criterion is mean square error.

   (a) *(15 points)* Prove that for a symmetric probability distribution on the input signal and an odd number of quantization levels, the center level is equal to zero and other levels are mirror images of each other.

   SOLUTION: *I just love general problems! Suppose we have a distribution $f_X(x)$ which is symmetric about some value v. That is, $f_X(v+x) = f_X(v-x)$. Then define the quantization levels as q_1 through q_N where $N = 2k+1$. The center level is then q_{k+1}. FINISH LATER*

   (b) *(15 points)* Find the optimal three level quantizer for x when

$$f_X(x) = \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

.

SOLUTION: *$q_2 = 0$ and $q_1 = -q_3$ by the first part. So $x_1 = q_1/2$ and $x_2 = q_3/2 = -q_1/2$. So, $x_1 = -x_2$. Thus, $q_2 = E[X|X \in (x_1, x_2)] = 0$ since the distribution is symmetric, so we're know that Loyd-Max is satisfied for q_2 and $x_1 = -x_2$. Now all that's left is determining x_1 and q_1.*

$$q_1 = E[X|X \in (-1, x_1)] = \frac{1}{p}\int_{-1}^{x_1} x(x+1)dx$$

where

$$p = \int_{-1}^{x_1}(x+1)dx = \frac{1}{2}(x+1)^2|_{-1}^{x_1} = \frac{1}{2}(x_1+1)^2$$

so

$$q_1 = \frac{2}{(x_1+1)^2}(x_1^3/3 + x_1^2/2 + 1/3 - 1/2)$$

or

$$6q_1 = \frac{1}{(x_1+1)^2}\left[x_1^2(4x_1+6) - 2\right] = 12x_1$$

which becomes

$$\left[x_1^2(4x_1+6) - 2\right] = 12x_1(x_1+1)^2$$

or Simplifying:

$$4x_1^3 + 9x_1^2 + 6x_1 + 1 = 0$$

which you can factor into

$$(x_1 + \frac{1}{4})(x_1+1)^2$$

so we have $x_1 = -\frac{1}{4}$ since the roots of -1 make no sense (it's a 3 level quantizer not a 1 level quantizer :).

If you blew the last factoring, no problem, you still get full credit if you had the right expressions – in practice you could use a numerical equation solver to get the proper roots. We then have $q_1 = -1/2$ so in total

$$Q(x) = \begin{cases} -1/2 & x \in (-1, \frac{-1}{4}) \\ 0 & x \in (\frac{-1}{4}, 1) \\ 1/2 & x \in (\frac{-1}{4}, \frac{1}{4}) \end{cases}$$

(c) *(10 points)* Let $y = x + c$ where x is distributed as in part 3b and c is a constant. Find the optimum three level quantizer for y.

SOLUTION: *Since* $E[Y|Y \in (x_i + c, x_{i+1} + c)] = E[X|X \in (x_i, x_{i+1})] + c = q_i + c$, *we have the exact same quantizer, just shifted over by* c.

4. *(60 points)* **Cora and the Wild Weevil:** Cora the communications engineer has been hired by We Wax Weevils Pest Control to find and destroy certain pests. A "weevil" makes a certain type of noise when munching what weevils munch on. Specifically, a weevil makes a known sound $s(t)$ during an observation interval $(0, T)$. To determine whether a weevil is present in an observation area, Cora deploys sensors which record $r_i = \int_0^T [s(t) + w(t)]\phi_i(t)dt$ where i is the sensor index, $\phi_i(t)$ is the sensor characteristic and $w(t)$ is Gaussian white noise with spectral intensity $N_0/2$. We define $E_s = \int_0^T s^2(t)dt$. Cora's job is to determine whether a weevil is present or not.

(a) *(10 points)* Suppose Cora has deployed exactly one sensor. What should $\phi_1(t)$ be to minimize the probability of error P_e where

$$P_e = (1-p)\text{Prob}(\text{say weevil}|\text{weevil not there}) + p\text{Prob}(\text{say no weevil}|\text{weevil there})$$

and p is the probability a weevil is present during the observation interval.

HINT: Use the matched filter result and adapt it to this receiver structure.

SOLUTION: *We have* $r_1 = A + n$ *where* $A = \int_0^T s(t)\phi_1(t)dt$ *if the weevil is present and zero if not, and* $n = \int_0^T w(t)\phi_1(t)dt$. n *is a Gaussian random variable since* $w(t)$ *is white and Gaussian. This is almost exactly our problem for which we derived the LRT (except there we used* $b = (-1, 1)$ *instead of as here we have* $b = (0, 1)$. *So to minimize the probability of error, we want to maximize A. A is an integral expression and we see that if we let* $h(t) = \phi_1(T - t)$ *we'd have our familiar LTI filter receiver. So, we then know that if we set* $\phi_1(t) = s(t)$ *we'll maximize A and hence minimize the probability of error.*

This sort of receiver is called a CORRELATOR receiver and is identical in performace to the matched filter receiver. The difference is that instead of using the (cumbersome, to me) LTI form, we simply multiple and integrate – something that's REALLY EASY with computers, ya know!

(b) *(10 points)* Suppose we let $\phi_1(t) = s(t)/\sqrt{E_s}$ and produce $r = b\sqrt{E_s} + n$ at the receiver where $b = 1, 0$ depending on whether the weevil is present or not respectively, and $n = \int_0^T w(t)\phi_1(t)dt$. Please provide a decision rule which minimizes the probability of error.

SOLUTION: *Pretty obvious given the previous part. We have a Gaussian centered at zero for r when the weevil is not there and centered at A when the weevil is there. So, if* $r > A/2$ *we scream WEEVIL and call in the napalm (maybe not so funny in this dangerous world we're living in these days) and if* $r < A/2$ *we scream NO weevil. If* $r = A$ *we can flip a coin since it does not affect the probability of error (because* $r = A/2$ *is a zero probability event).*

What is A? $A = \int_0^T s(t)\phi_1(t)dt = \sqrt{E_s}$.

(c) *(10 points)* What is the minimum probability of error if $p = 1$? How about $p = 0$. Assuming $p = 1 - p = 1/2$, please derive an expression for the probability of error which depends only on E_s and N_0.

SOLUTION: *If $p = 1$ we know the weevil is always there and we say so (zero probability of error). Same idea for $p = 0$. For $p = 1/2$ we get an error if the weevil is there and we say it's not. The probability of that conditional event is*

$$P_{e|weevil\ there} = \int_{-\infty}^{\sqrt{E_s}2} \frac{1}{\sqrt{\pi N_0}} e^{-(x-\sqrt{E_s})^2/N_0} dx$$

since the variance of n is $N_0/2$. We change variables to obtain

$$P_{e|weevil\ there} = \int_{-\infty}^{-\sqrt{E_s/2N_0}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{\sqrt{E_s/2N_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

We get the same thing for $P_{e|weevil\ not\ there}$ and since both weevil there and weevil not there are probability 1/2 events,

$$P_e = \int_{\sqrt{E_s/2N_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

(d) *(20 points)* Suppose Cora can now use two sensors each with characteristic $\phi_i(t)$ and with the property that $\int_0^T \phi_i(t)\phi_j(t)dt = \delta_{ij}$ where δ_{ij} is the usual discrete delta function. Further, define $s_1 = \int_0^T s(t)\phi_1(t)dt \neq 0$ and $s_2 = \int_0^T s(t)\phi_2(t)dt \neq 0$. Derive a receiver which minimizes the probability of error using the observed values $r_i = \int_0^T r(t)\phi_i(t)dt$. Again, assume that $p = 1/2$.

HINTS: The received signal is a point $\mathbf{r} = (r_1, r_2)$. Think of the conditional probability distribution on $\mathbf{r}$ when $b = 1$ and $b = 0$ and adapt the scalar results to the vector case. If $\int_0^T \phi_1(t)\phi_2(t)dt = 0$ what can we say about the random variables $\int_0^T w(t)\phi_i(t)dt$? A sketch will be helpful.

SOLUTION: *Suppose there is no noise. If the Weevil is present, then you get a point $\mathbf{r} = (s_1, s_2)$. If not, then $\mathbf{r} = (0, 0)$. Now, we also have $r_i = bs_i + n_i$ where $b = 0, 1$ and $n_i = \int_0^T \phi_i(t)w(t)dt$. The n_i are jointly Gaussian and because the $\phi_i(t)$ are orthonormal, they have equal variance ($N_0/2$) and are INDEPENDENT. So, the conditional distribution given the Weevil is present is a 2-D Gaussian product distribution centered on $\mathbf{s} = (s_1, s_2)$. And if the Weevil is not present it's the same distribution centered on $(0, 0)$.*

More formally:

$$f_{\mathbf{R}|weevil!}(\mathbf{r}|weevil!) = \frac{1}{\pi N_0} e^{-\frac{(r_1-s_1)^2+(r_2-s_2)^2}{N_0}} = \frac{1}{\pi N_0} e^{-\frac{|\mathbf{r}-\mathbf{s}|^2}{N_0}}$$

where $\mathbf{s} = (s_1, s_2)$. Likewise we have

$$f_{\mathbf{R}|weevil!}(\mathbf{r}|weevil!) = \frac{1}{\pi N_0} e^{\frac{|\mathbf{r}|^2}{N_0}}$$

We then form our LRT

$$\frac{f_{\mathbf{R}|weevil!}(\mathbf{r}|WEEVIL!)}{f_{\mathbf{R}|no\ weevil!}(\mathbf{r}|no\ weevil!)} = e^{(|\mathbf{r}|^2-|\mathbf{r}-\mathbf{s}|^2)/N_0} \quad \underset{No\ Weevil}{\overset{Weevil}{\gtrless}} \quad 1$$

We can look at this expression and notice that the line for which the ratio equals 1 is the line which perpendicularly bisects the line between the points $(0,0)$ *and* $\mathbf{s}$*. Or if our intuition is not honed we can proceed formally. That is, we note that* $|\mathbf{r}|^2 = \mathbf{r}^\top\mathbf{r}$ *and* $|\mathbf{r}-\mathbf{s}|^2 = \mathbf{r}^\top\mathbf{r} + \mathbf{s}^\top\mathbf{s} - 2\mathbf{s}^\top\mathbf{r}$ *so that*

$$e^{(2(\mathbf{r}-\frac{\mathbf{s}}{2})^\top\mathbf{s})/N_0} \quad \underset{\textit{No Weevil}}{\overset{\textit{Weevil}}{\gtrless}} \quad 1$$

We can take the log (monotone function does not bother either side of the inequality) and we get

$$(\mathbf{r}-\frac{\mathbf{s}}{2})^\top\mathbf{s} \quad \underset{\textit{No Weevil}}{\overset{\textit{Weevil}}{\gtrless}} \quad 0$$

We can then define a new "observation vector" $\mathbf{z} = \mathbf{r} - \frac{\mathbf{s}}{2}$ *and have*

$$\mathbf{z}^\top\mathbf{s} \quad \underset{\textit{No Weevil}}{\overset{\textit{Weevil}}{\gtrless}} \quad 0$$

and what we see is this. We're PROJECTING the observation $\mathbf{z}$ *onto the vector* $\mathbf{s}$*. If the projection is positive (this means the observed point is closer to* $\mathbf{s}$ *than it is to* $(0,0)$*), we say the weevil is there. If not, then we say the weevil is not there. This is your first taste of signal space and what we've done is derive DISTANCE BASED DECODING. If the point you see is near what you know to be a signal point, you declare that signal point!!!!*

(e) *(10 points)* When the infamous Wild Weevil is present, it can present a variety of sounds $s(t)$. Specifically, on any given interval, $s(t) = \phi_1(t)\cos X + \phi_2(t)\sin X$ where X is a uniform random variable on $(0, 2\pi)$. Sketch the decision regions for optimally detecting the Wild Weevil (but do not evaluate the probability of error, nor attempt to derive exact thresholds). Again, a sketch will be helpful.

SOLUTION: *In the absence of noise, if we look at all the possible points* (r_1, r_2) *associated with the signal corresponding to the presence of the the Wild Weevil, we see that it forms a circle at radius* 1 *around the origin. If the Wild Weevil is not there, then we have a single point at the origin. So, owing to the circular symmetry, Cora's detector will have to look at how far the received point is from the origin. If it's beyond some distance, then Cora declares the Wild Weevil present. If not, then Cora says no Weevil present. The problem corresponds to the digital communications problem when we don't know the phase of an incoming signal (hence the circular symmetry). Carefully deriving the critical radius is not very hard, but seemed nasty to ask you to do as the very last problem.*

Hope you enjoyed this exam!!!!