Both Cosine and Sine signals are called Sinusoidal signals or simply Sinusoids. Cosine can be related to Sine and vice versa.

$$\boxed{\sin(\omega t) = \cos(\omega t - 90^{0})}$$

Sinusoidal Signals

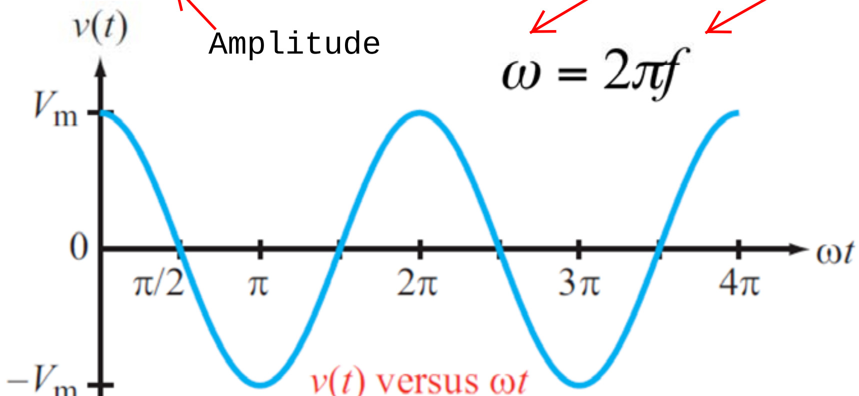
Omega is the angular frequency in radians/sec

f is the cyclic frequency in cycles/sec or Hertz

$$v(t) = V_m \cos(\omega t + \phi)$$

Amplitude

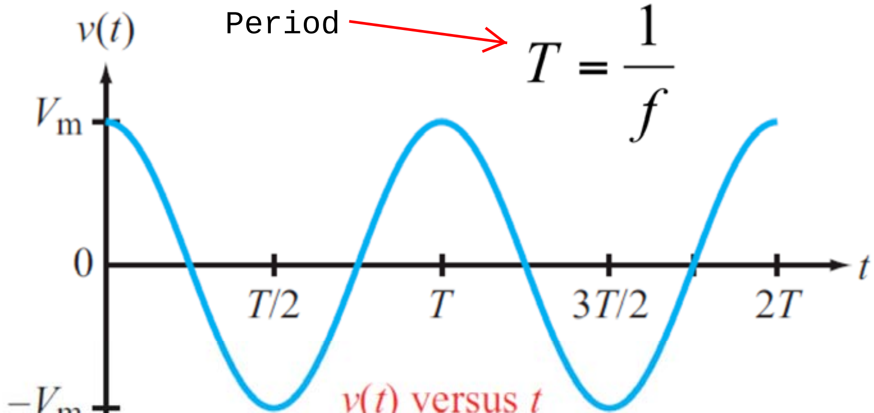
$$\omega = 2\pi f$$



for ϕ =0

Period

$$T = \frac{1}{f}$$



for ϕ =0

Useful relations

$$\sin x = \pm\cos(x \mp 90^\circ)$$
$$\cos x = \pm\sin(x \pm 90^\circ)$$
$$\sin x = -\sin(x \pm 180^\circ)$$
$$\cos x = -\cos(x \pm 180^\circ)$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2\sin x \cos y = \sin(x + y) + \sin(x - y)$$
$$2\cos x \cos y = \cos(x + y) + \cos(x - y)$$

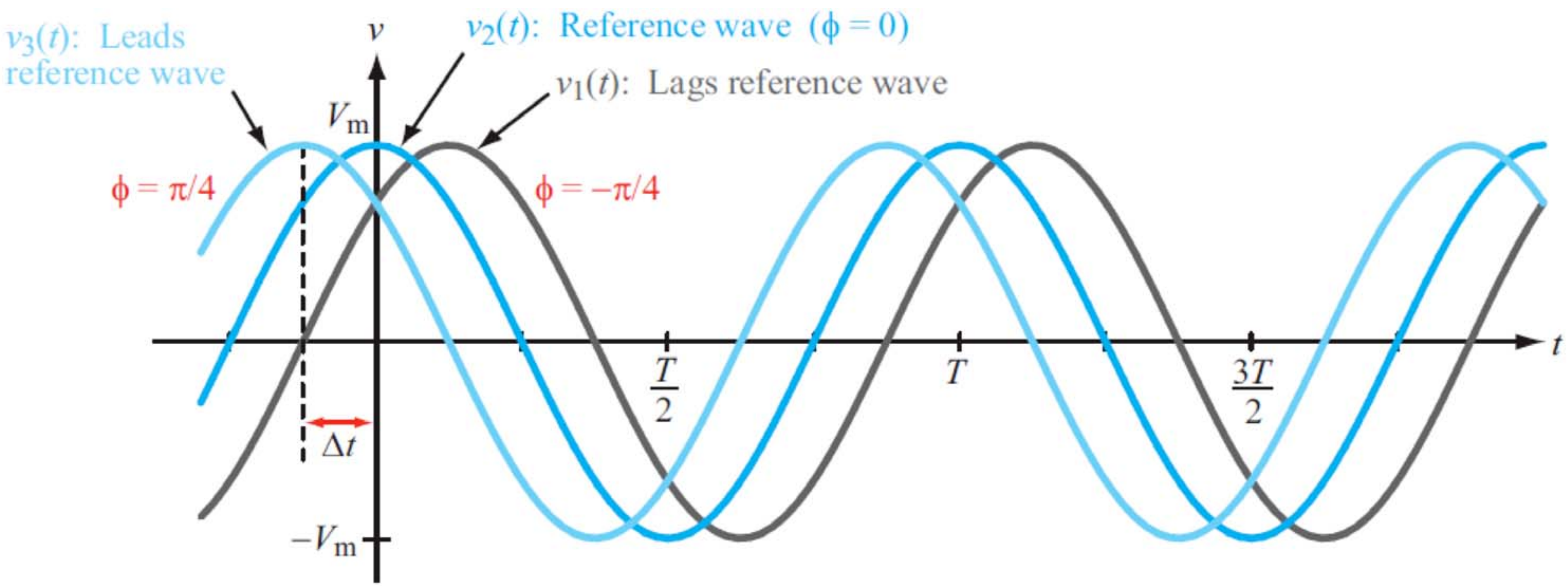
Phase Lead/Lag

$$v_1(t) = V_m \cos\left(\frac{2\pi t}{T} - \frac{\pi}{4}\right), \tag{7.10a}$$

$$v_2(t) = V_m \cos\frac{2\pi t}{T} \quad \text{(Reference waveform with } \phi = 0\text{)}, \tag{7.10b}$$

and

$$v_3(t) = V_m \cos\left(\frac{2\pi t}{T} + \frac{\pi}{4}\right). \tag{7.10c}$$



Both Sine and Cosine signals are called sinusoids.
They are also called AC signals as they are common in many ways.

- Sinusoidal input is common in electronic circuits
- Any time-varying periodic signal can be represented by a series of sinusoids (Fourier Series)

Objective: To determine the steady state response of a linear circuit to ac signals

What do we mean by steady state response of a circuit?

Principles of Electrical Engineering I

Motivation to steady state analysis

Consider the RC circuit of Figure 1.
The equation that inter connects the input v_{in} and the output v_o is

$$v_{in} = RC\frac{dv_o}{dt} + v_o.$$

The complete solution of the above equation is

$$v_o(t) = v_o(0)e^{-\frac{t}{RC}} + \frac{1}{RC}\int_0^t e^{-\frac{t-\tau}{RC}} v_{in}(\tau)d\tau$$

First Order RC Circuit:

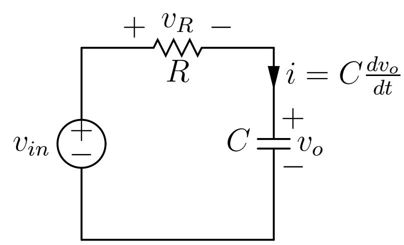


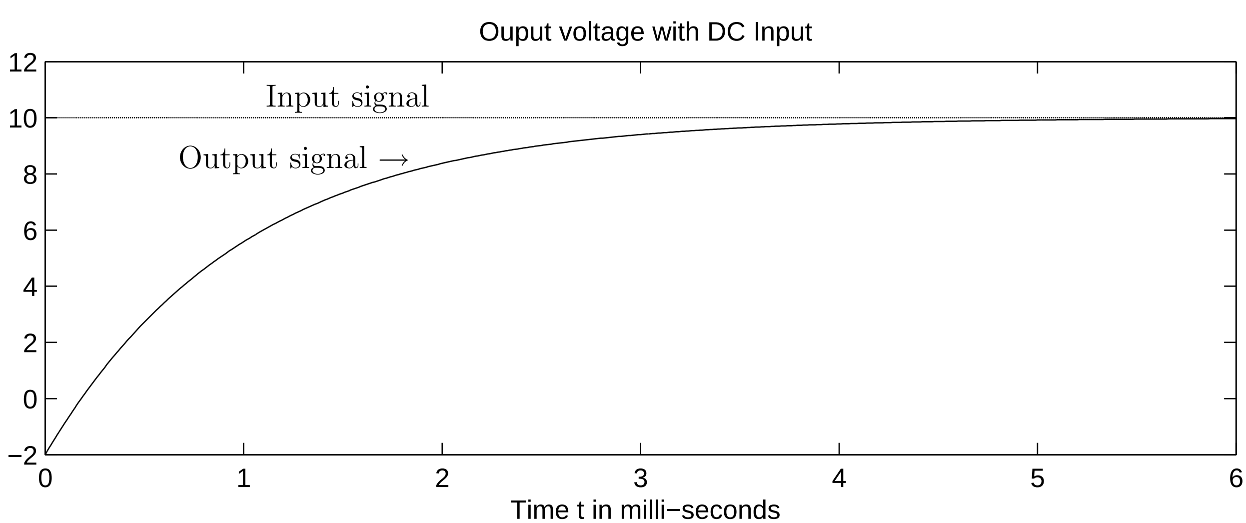
Figure 1

where $v_o(0)$ is the initial value of $v_o(t)$ at time $t = 0$. The solution given above is said to be the complete solution (or complete response) which for all typical inputs has two parts, transient response and steady state response. We observe that the transient response is that part of the complete response which dies out as time t progresses to infinity, where as the steady state response is that part of the complete response which predominates as time t progresses to infinity.

Constant input signal (DC input): Let $v_{in} = E$ where E is a constant (DC input). The complete solution then simplifies to

$$v_o(t) = \underbrace{[v_o(0) - E]\, e^{-\frac{t}{RC}}}_{\boxed{\text{Transient response}}} + \underbrace{E}_{\boxed{\text{Steady State response}}}.$$

Figure given below shows the DC input as well as the complete output. To draw the graph, we used $E = 10$ Volts, $v_o(0) = -2$ Volts, $R = 1\Omega$ and $C = 10^{-3}$ Farads. In this example, transient part dies out in about 4 or 5 milli-seconds, and for $t \geq 5$ milli-seconds the output is in steady state, i.e. it follows the pattern of the input which for DC is a constant.



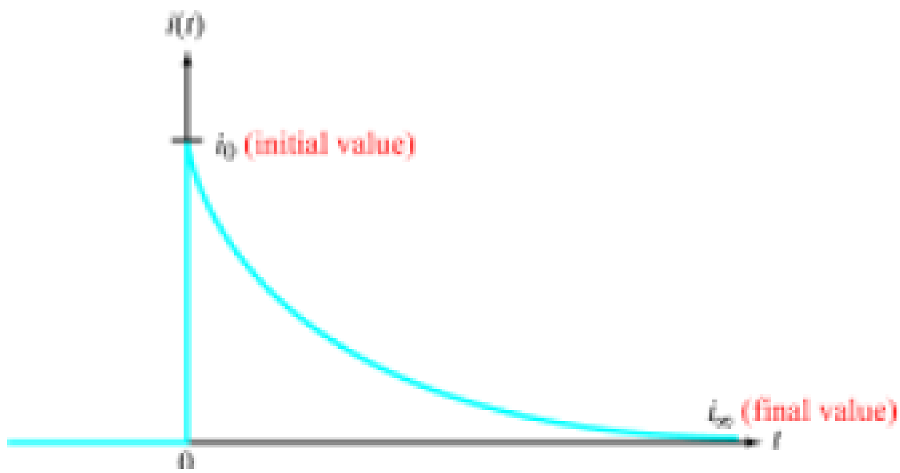
Exponential Decay

t	$e^{-t/\tau}$
0	1
τ	0.36788
2τ	0.13534
3τ	0.049787
4τ	0.018316;
5τ	0.0067379

Consider a function,

$$i(t) = i(0)e^{-\frac{t}{\tau}} \quad \text{for} \quad t \geq 0.$$

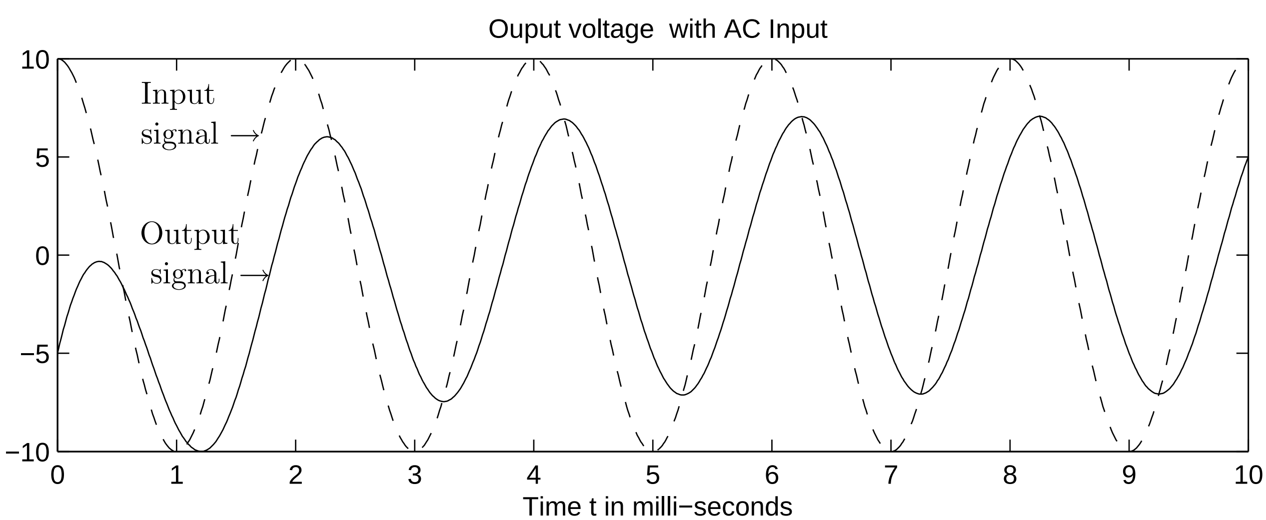
Initial Value $i(0) = 1$
Final Value $i(\infty) = 0$
Rate of decay depends on τ which is called *Time Constant*.
In a duration of $t = \tau$, the value of $i(t)$ decreases by a factor 0.36788.
In a duration of $t = 5\tau$, the value of $i(t)$ would for all practical purposes decreases to zero.
Slope of $i(t)$ at $t = 0$ is $-\frac{1}{\tau}$.
The entire area under the curve $e^{-\frac{t}{\tau}}$ is τ.



Alternating input signal (AC input): Let $v_{in} = A\cos(\omega t)$ where ω is the angular frequency in radians per second and A is the amplitude of the sinusoidal voltgae in Volts (AC input). The $\boxed{\text{complete solution}}$ then simplifies to

$$\begin{aligned} v_o(t) &= \underbrace{\left[v_o(0) - \frac{A}{1+\omega^2R^2C^2}\right]e^{-\frac{t}{RC}}}_{\boxed{\text{Transient response}}} + \underbrace{\frac{A}{1+\omega^2R^2C^2}[\cos(\omega t) + \omega RC\sin(\omega t)]}_{\boxed{\text{Steady State response}}} \\ &= \underbrace{\left[v_o(0) - \frac{A}{1+\omega^2R^2C^2}\right]e^{-\frac{t}{RC}}}_{\boxed{\text{Transient response}}} + \underbrace{\frac{A}{\sqrt{1+\omega^2R^2C^2}}[\cos(\omega t - \theta)]}_{\boxed{\text{Steady State response}}} \text{ where } \theta = tan^{-1}(\omega RC). \end{aligned}$$

Figure given below shows the AC input as well as the complete output. To draw the graph, we used $A = 10$ Volts, $v_o(0) = -5$ Volts, $\omega = 10^3\pi$ radians per second, $R = 1\Omega$, and $C = 10^{-3}$ Farads. In this example, transient part dies out in about 5 milli-seconds, and for $t \geq 5$ milli-seconds the output is in steady state, i.e. it follows the pattern of the input which for AC is sinusoidal. It is easy to see from the above expression for $v_o(t)$ that when the input is a sinusoidal signal of certain frequency, the output is also a sinusoidal signal of the same frequency, however with a different amplitude and phase.



Our interest often is to determine the **steady state voltage and current values** for a given input signal. That is, we are often interested in **steady state analysis** of a circuit. In this regard, we often consider input signals as either DC signals or AC signals. There is a fundamental reason why we consider so. *It turns out that any practical signal can be expressed as a sum of a DC signal and sinusoids of different amplitudes, frequencies, and phase angles.* Thus, if we know how to analyze a circuit when inputs are DC signals and sinusoidal signals, then in principle we can analyze a circuit for any given input signal. In what follows we study techniques of DC steady state analysis and sinusoidal steady state analysis.

Let us also remark at this time that a DC signal can be considered as an AC sinusoidal signal having zero frequency. Although a DC signal can be considered as an AC sinusoidal signal, the DC steady state analysis is much simpler to do directly than trying to obtain the analysis as a limiting case of a sinusoidal steady state analysis as frequency tends to zero.

Principles of Electrical Engineering I

Motivation to introduce phasors

Consider the RC circuit of Figure 1.
The equation that inter connects the input v_{in} and the output v_o is

$$v_{in} = RC\frac{dv_o}{dt} + v_o.$$

Let $v_{in} = A\cos(\omega t)$ where ω is the angular frequency in radians per second and A is the amplitude of the sinusoidal voltgae in Volts (AC input).

First Order RC Circuit:

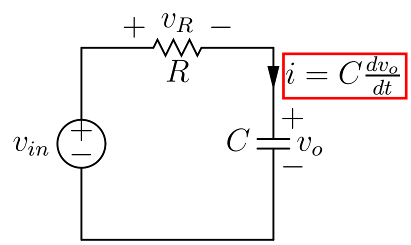


Figure 1

Our interest is to determine the steady state output $v_o(t)$. As we discussed earlier, the steady state output is a sinusoidal signal of the same angular frequency ω, however it could have different amplitude and phase angle. So let us assume that in steady state

$$v_o(t) = A_o\cos(\omega t + \theta_o).$$

Then, the voltage across R is given by

$$v_R(t) = RC\frac{dv_o}{dt} = -A_o\omega RC\sin(\omega t + \theta_o) = A_o\omega RC\cos(\omega t + \theta_o + 90^0).$$

As seen clearly, both $v_o(t)$ and $v_R(t)$ are sinusoidal signals, and these must be added and equated to the input sinusoid $A\cos(\omega t)$ in order to determine A_o and θ_o. That is, A_o and θ_o satisfy the equation

$$A_o\omega RC\cos(\omega t + \theta_o + 90^0) + A_o\cos(\omega t + \theta_o) = A\cos(\omega t).$$

We observe that the algebra involved in determining A_o and θ_o is simply the algebra involved in adding sinusoidal signals of the same frequency but different amplitudes and phase angles.
The above observation is **not surprising** because, as we learned so far, all the circuit analysis is based on Kirchoff's laws, KCL and KVL. As we should know thoroughly by now, KCL simply says that the algebraic sum of all currents at a node is zero, while KCL simply says that the algebraic sum of all voltages along a closed path is zero. If all the currents and voltages are sinusoidal signals each having the same frequency but possibly different amplitudes and phases, then the algebra involved in utilizing KCL and KVL equations is merely the algebra of adding such sinusoids.
To understand the algebra involved in adding sinusoids, let us consider next the addition of two real sinusoids.

Example of adding two real sinusoids: Determine the sum of two sinusoidal signals $x_1(t)$ and $x_2(t)$. That is, determine $x(t) = x_1(t) + x_2(t)$, where

$$x_1(t) = A_1 \cos(\omega t + \theta_1) \quad \text{and} \quad x_2(t) = A_2 \cos(\omega t + \theta_2).$$

By using trigonometry, we can rewrite

$$\begin{aligned} A_1 \cos(\omega t + \theta_1) &= A_1 \cos(\theta_1)\cos(\omega t) - A_1 \sin(\theta_1)\sin(\omega t) \\ A_2 \cos(\omega t + \theta_2) &= A_2 \cos(\theta_2)\cos(\omega t) - A_2 \sin(\theta_2)\sin(\omega t). \end{aligned}$$

Thus, we note that

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) \\ x(t) &= A_1 \cos(\omega t + \theta_1) + A_2 \cos(\omega t + \theta_2) \\ &= \boxed{[A_1 \cos(\theta_1) + A_2 \cos(\theta_2)]}\cos(\omega t) - \boxed{[A_1 \sin(\theta_1) + A_2 \sin(\theta_2)]}\sin(\omega t) \\ &= A\cos(\theta)\cos(\omega t) - A\sin(\theta) sin(\omega t) \\ &= A\cos(\omega t + \theta), \end{aligned}$$

where

$$\boxed{A_1 \cos(\theta_1) + A_2 \cos(\theta_2) = A\cos(\theta) \quad \text{and} \quad A_1 \sin(\theta_1) + A_2 \sin(\theta_2) = A\sin(\theta).}$$

The above equations together can be rewritten in conventional polar coordinate notation of adding two vectors as

$$\boxed{A\angle\theta = A_1\angle\theta_1 + A_2\angle\theta_2.} \qquad (*)$$

The above analysis shows that the sum of two real sinusoids of the same frequency is another real sinusoid of the same frequency whose amplitude and phase angle is given by the above equation. The above equation does not depend at all on time t. It merely depends on the amplitudes and phase angles of the given real sinusoids.

Important Observation: The sum of two real sinusoids is another real sinusoid whose amplitude and phase are given by equation (*). A simple study of equation (*) reveals that it is simply the sum of two vectors yielding a resultant vector. In Electrical Engineering, a vector derived from the amplitude and phase of a sinusoid is called a **phasor**. Thus, the equation (*) represents indeed a sum of two phasors to yield another phasor.

Let us **emphasize** once again that we call
$A_1\angle\theta_1$ as the phasor of $A_1\cos(\omega t + \theta_1)$,
$A_2\angle\theta_2$ as the phasor of $A_2\cos(\omega t + \theta_2)$, and
$A\angle\theta$ as the phasor of $A\cos(\omega t + \theta)$.
To add

$$A_1 \cos(\omega t + \theta_1) \quad \text{and} \quad A_2 \cos(\omega t + \theta_2),$$

we add two phasors

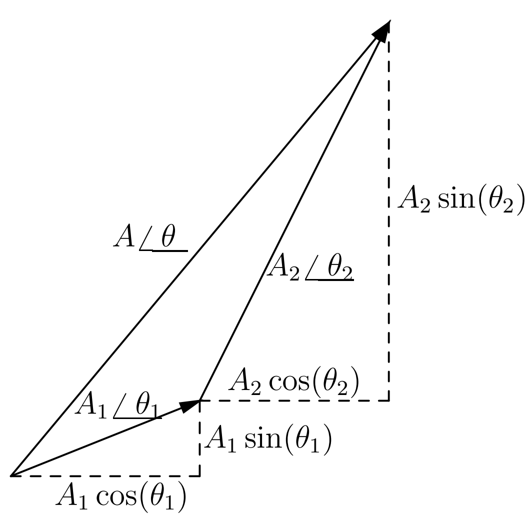
$$A_1\angle\theta_1 + A_2\angle\theta_2,$$

to get

$$A\angle\theta.$$

This phasor $A\angle\theta$ is converted back into time domain as

$$A\cos(\omega t + \theta).$$



Phasor Addition

Phasor and Inverse Phasor Transformations

Two real sinusoids

$$A_1 \cos(\omega t + \theta_1) \quad \text{and} \quad A_2 \cos(\omega t + \theta_2)$$

when added results in another real sinusoid

$$A \cos(\omega t + \theta).$$

In order to obtain the amplitude A and the phase angle θ of the resulting real sinusoid, we do the following arithmetic,

$$A_1 \angle \theta_1 + A_2 \angle \theta_2 = A \angle \theta.$$

We call
$A_1 \angle \theta_1$ as the phasor of $A_1 \cos(\omega t + \theta_1)$,
$A_2 \angle \theta_2$ as the phasor of $A_2 \cos(\omega t + \theta_2)$, and
$A \angle \theta$ as the phasor of $A \cos(\omega t + \theta)$.

The above discussion leads us to the following *Phasor and Inverse Phasor Transformations.*

When we use $\cos(\omega t)$ as the reference sinusoid, we have the following transformation from *time domain* to *phasor domain*:

$$\boxed{\begin{array}{ccc} \textit{Time domain} & & \textit{Phasor domain} \\ A\cos(\omega t + \theta) & \Rightarrow & Ae^{j\theta} = A\angle\theta. \end{array}}$$

We will
discuss this soon

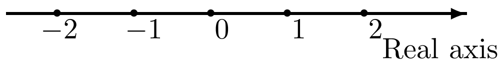
When we use $\cos(\omega t)$ as the reference sinusoid, we have the following inverse transformation from *phasor domain* to *time domain*:

$$\boxed{\begin{array}{ccc} \textit{Phasor domain} & & \textit{Time domain} \\ Ae^{j\theta} = A\angle\theta & \Rightarrow & A\cos(\omega t + \theta). \end{array}}$$

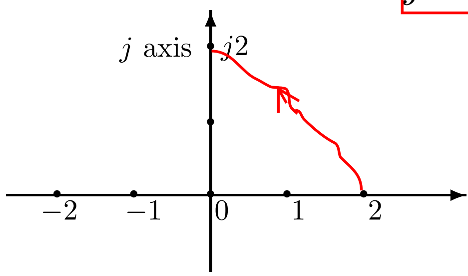
In order to learn more about phasors and phasor domain analysis, we need to review complex numbers.

Review of Complex Numbers

- Real numbers are represented along the x-axis.



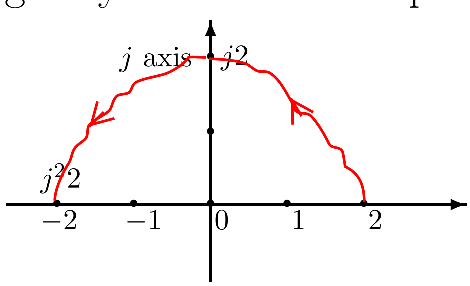
- j is an 'operator' that rotates counterclockwise by 90°.
  What is an 'operator'?
  An 'operator' is something that works on a 'number'
  (or a function) to generate another 'number' (or a function).
  For example, take a real number 2.
  Then $j2$ rotates 2 by 90° counterclockwise.
  Thus $j2$ is a number along the y-axis.
  Hence y-axis can be called as the j axis.



- j^2 is an operator j applied twice,
  that is j^2 is a counter-clock-wise rotation of 180°.

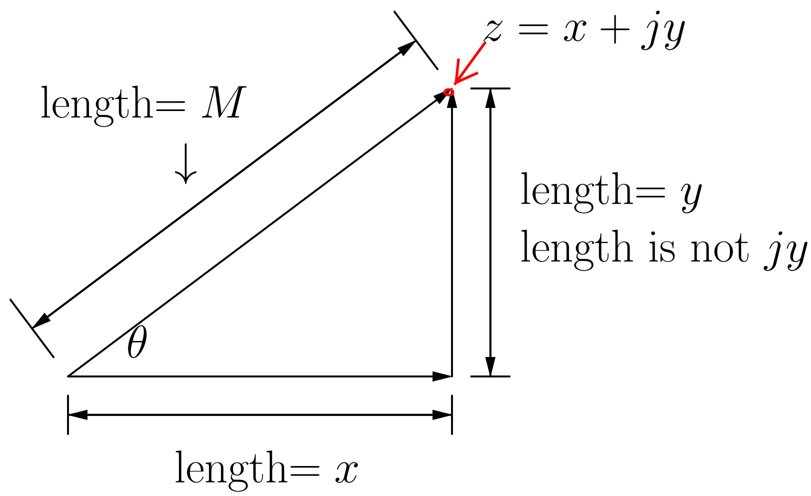
- Take a real positive number, say 2.
  Then $j^2 2$ rotates 2 counterclockwise by 180°.
  Thus $j^2 2 = -2$. In other words, $j^2 = -1$.
  Hence, j has the interpretation of being $\sqrt{-1}$.
  Because of this $j = \sqrt{-1}$ is called an imaginary number.
  Thus, y-axis is called by two names,
  (1) the j axis and (2) the imaginary axis.
  There is nothing imaginary about the axis; it is just a name,
  because imaginary numbers are represented along it.

Notation i is used in math instead of j



- A number $x + jy$ is called a complex number.
- A complex number can be represented in three ways:
  - Rectangular form (in rectangular coordinates x and y).
  - Polar form as $M\angle\theta$, that is in polar coordinates, a magnitude M and an angle θ.
  - In the exponential form as $Me^{j\theta}$.

  One can convert a rectangular form to a polar form and vice versa by simply looking at the triangle, see figure below.



The relationship between the polar and rectangular coordinates of a complex number z is as shown below:

$$\begin{aligned} z = M\angle\theta &= Me^{j\theta} = x + jy \\ x &= M\cos(\theta), \quad y = M\sin(\theta), \\ \text{Absolute Value} \quad |z| = M &= \sqrt{x^2 + y^2}, \quad \text{and} \quad \tan(\theta) = \frac{y}{x}. \end{aligned}$$

- Euler's theorem states that $Me^{j\theta} = M\cos(\theta) + jM\sin(\theta)$. Thus $M\angle\theta$ is same as $Me^{j\theta}$.

Addition: Addition of two complex numbers is straight forward in rectangular form.

$$(a + jb) + (c + jd) = (a + c) + j(b + d).$$

Subtraction: Subtraction of two complex numbers is straight forward in rectangular form.

$$(a + jb) - (c + jd) = (a - c) + j(b - d).$$

Multiplication: We first show multiplication of two complex numbers in rectangular form.

$$\begin{aligned}(a + jb)(c + jd) &= ac + jad + jbc + j^2bd \\ &= (ac - bd) + j(ad + bc).\end{aligned}$$

$\boxed{j^2 = -1}$

We show next multiplication of two complex numbers in polar form. Indeed

$$\begin{aligned}(M_1e^{j\theta_1})(M_2e^{j\theta_2}) &= M_1M_2e^{j(\theta_1+\theta_2)} \\ &= M_1M_2\angle\underline{\theta_1 + \theta_2}.\end{aligned}$$

Division: We first show division of two complex numbers in rectangular form.

$$\begin{aligned}\frac{a + jb}{c + jd} &= \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} \\ &= \frac{ac + jbc - jad - j^2bd}{c^2 + jdc - jdc - j^2d^2} \\ &= \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}.\end{aligned}$$

$\boxed{j^2 = -1}$

We show next division of two complex numbers in polar form. Indeed

$$\frac{M_1 e^{j\theta_1}}{M_2 e^{j\theta_2}} = \frac{M_1}{M_2} e^{j(\theta_1-\theta_2)} = \frac{M_1}{M_2} \angle \theta_1 - \theta_2.$$

If z is a complex number given by $z = M\angle\theta = Me^{j\theta} = \boxed{x + jy},$ then its conjugate (denoted by z^*) is given by

Complex conjugate

$$z^* = M\angle -\theta = Me^{-j\theta} = \boxed{x - jy}.$$

Also,

$$zz^* = M^2 = x^2 + y^2, \quad z + z^* = 2x, \quad z - z^* = j2y.$$

Thus

$$e^{j\theta} + e^{-j\theta} = 2\cos(\theta) \quad \text{and} \quad e^{j\theta} - e^{-j\theta} = 2j\sin(\theta).$$

Example: We note that

$$\frac{1}{j} = \frac{j}{j^2} = -j, \text{ and } j = -\frac{1}{j}.$$

Example: We note that

$$e^{j\frac{\pi}{2}} = e^{j90^{\circ}} = j,$$

$$e^{-j\frac{\pi}{2}} = e^{-j90^{\circ}} = -j.$$

Note that $-j$ is a counterclockwise rotation of -90° or a clockwise rotation of 90°.

Example: Note the rectangular to polar form of the following four numbers:

Complex conjugates

$a = 0.5 + j0.866 = 1\angle 60^{\circ}$

$b = -0.5 + j0.866 = 1\angle 120^{\circ}$

Complex conjugates

$c = -0.5 - j0.866 = 1\angle -120^{\circ}$

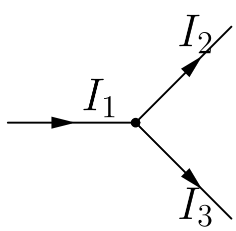
$d = 0.5 - j0.866 = 1\angle -60^{\circ}$

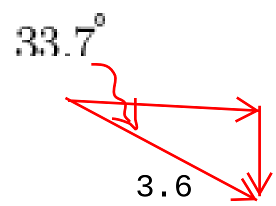
Note: Since all these four numbers in rectangular form are distinct from one another, the corresponding polar forms must be distinct from one another as well.

Example: The phasor currents in two elements connected in series are given by $I_1 = M\angle 60^{\circ}$ and $I_2 = x + j2.771$. Find the values of x and M.

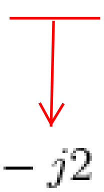
We note that two complex numbers are equal if and only if their real parts are equal and the imaginary parts are equal.
Also, I_1 and I_2 must equal each other as the given two elements are connected in series.
Moreover, $I_1 = M\angle 60^{\circ} = M\cos(60^{\circ}) + jM\sin(60^{\circ})$,
and $\boxed{I_2 = x + j2.771.}$
The equality of I_1 and I_2 implies that
$\boxed{x = M\cos(60^{\circ}) \text{ and } M\sin(60^{\circ}) = 2.771.}$
The later implies that $M = \frac{2.771}{\sin(60^{\circ})} = 3.2$.
This enables us to calculate $x = M\cos(60^{\circ}) = 3.2\cos(60^{\circ}) = 1.6$.
Example: In a particular circuit at a particular node, the phasor currents I_1, I_2, and I_3 are given by

$$\begin{aligned} I_1 &= M\angle\theta \\ I_2 &= 2\angle -90^{\circ} \\ I_3 &= 3.6\angle -33.7^{\circ}. \end{aligned}$$

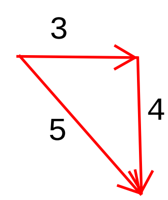




Find the magnitude M and angle θ.
As we shall see, the KCL equation at the node must be true even for phasor currents. Thus



$$\begin{aligned} I_1 &= I_2 + I_3 = 2\angle -90^{\circ} + 3.6\angle -33.7^{\circ} \\ &= 0 - j2 + 3.6\cos(33.7^{\circ}) - j3.6\sin(33.7^{\circ}) \\ &= 0 - j2 + 3 - j2 = 3 - j4 = 5\angle -53.13^{\circ}. \end{aligned}$$



Thus, $M = 5$ and $\theta = -53.13^{\circ}$.

$$\boxed{\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5}$$

$$\boxed{\theta = \tan^{-1}(\tfrac{-4}{3}) = -53.13^{\circ}}$$

Example: In any circuit, the phasor voltage V and phasor current I of an element are related by $V = IZ$ where Z is called the impedance of the element.
If $I = 26\angle{-120.5^\circ}$ and $Z = 3 + j4$, find the voltage V.

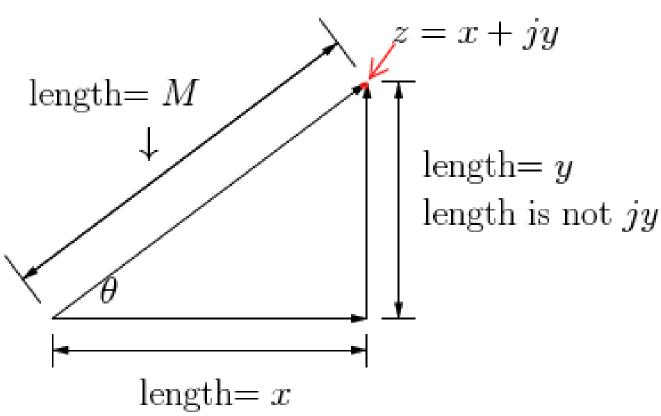
$$\begin{aligned} V &= IZ = 26\angle{-120.5^\circ}(3 + j4) = \\ &= 26\angle{-120.5^\circ}\ 5\angle{53.1^\circ} \\ &= 130\angle{-67.4^\circ} \\ &= 130\cos(-67.4^\circ) + j130\sin(-67.4^\circ) \\ &= 50 - j120 \text{ volts.} \end{aligned}$$

130

Example: In any circuit, the phasor voltage V and phasor current I of an element are related by $I = VY$ where Y is called the admittance of the element. If $I = 136 - j8$ and $V = 900 - j2100$, find the admittance Y.

$$\begin{aligned} Y &= \frac{I}{V} = \frac{136 - j8}{900 - j2100} = \frac{136.24\angle{-3.37^\circ}}{2284.6\angle{-66.8^\circ}} \\ &= \frac{136.24}{2284.6}\angle{-3.37^\circ + 66.8^\circ} \\ &= 59.63\angle{63.43^\circ} \text{ milli mhos.} \end{aligned}$$

$$\begin{aligned} 136 - j8 &= \sqrt{136^2 + 8^2}\angle{\tan^{-1}}\left(\frac{-8}{136}\right) \\ &= \sqrt{18496 + 64}\angle{-3.37^\circ} \\ &= 136.24\angle{-3.37^\circ} \end{aligned}$$



You cannot
forget this

Properties of complex numbers.

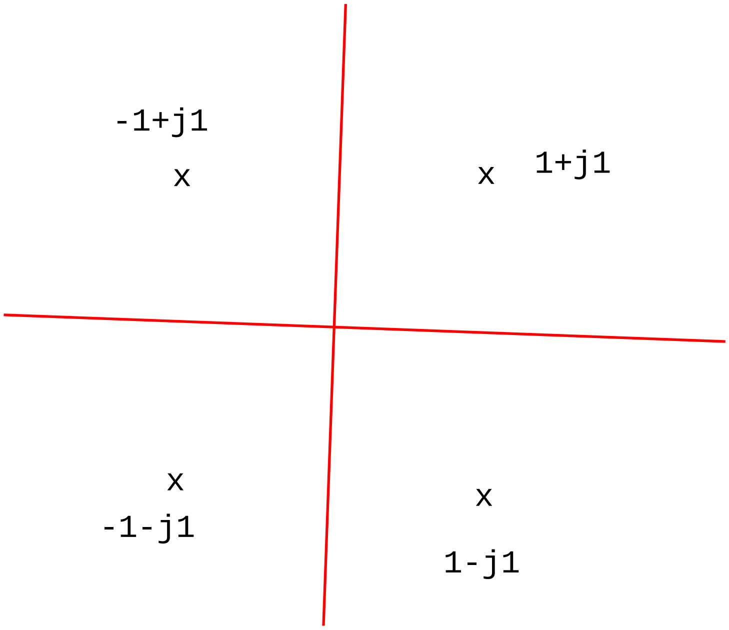
Euler's Identity: $e^{j\theta} = \cos\theta + j\sin\theta$	
$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$	$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$
$\mathbf{z} = x + jy = \lvert\mathbf{z}\rvert e^{j\theta}$	$\mathbf{z}^* = x - jy = \lvert\mathbf{z}\rvert e^{-j\theta}$
$x = \Re\mathfrak{e}(\mathbf{z}) = \lvert\mathbf{z}\rvert\cos\theta$	$\lvert\mathbf{z}\rvert = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2 + y^2}$
$y = \Im\mathfrak{m}(\mathbf{z}) = \lvert\mathbf{z}\rvert\sin\theta$	$\theta = \tan^{-1}(y/x)$
$\mathbf{z}^n = \lvert\mathbf{z}\rvert^n e^{jn\theta}$	$\mathbf{z}^{1/2} = \pm\lvert\mathbf{z}\rvert^{1/2} e^{j\theta/2}$
$\mathbf{z}_1 = x_1 + jy_1$	$\mathbf{z}_2 = x_2 + jy_2$
$\mathbf{z}_1 = \mathbf{z}_2$ iff $x_1 = x_2$ and $y_1 = y_2$	$\mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$
$\mathbf{z}_1\mathbf{z}_2 = \lvert\mathbf{z}_1\rvert\lvert\mathbf{z}_2\rvert e^{j(\theta_1+\theta_2)}$	$\dfrac{\mathbf{z}_1}{\mathbf{z}_2} = \dfrac{\lvert\mathbf{z}_1\rvert}{\lvert\mathbf{z}_2\rvert} e^{j(\theta_1-\theta_2)}$
$-1 = e^{j\pi} = e^{-j\pi} = 1\underline{/\pm 180^\circ}$	
$j = e^{j\pi/2} = 1\underline{/90^\circ}$	$-j = e^{-j\pi/2} = 1\underline{/-90^\circ}$
$\sqrt{j} = \pm e^{j\pi/4} = \pm\dfrac{(1+j)}{\sqrt{2}}$	$\sqrt{-j} = \pm e^{-j\pi/4} = \pm\dfrac{(1-j)}{\sqrt{2}}$

$$\frac{1}{j} = -j$$

$$j = 1\underline{/90^\circ}$$

$$-j = 1\underline{/-90^\circ}$$

$$j^2 = -1 = 1\underline{/180^\circ} = 1\underline{/-180^\circ}$$



-1+j1
1+j1
-1-j1
1-j1

Phasor Domain

A *domain transformation* is a mathematical process that converts a set of variables from their domain into a corresponding set of variables defined in another domain.

1. The phasor-analysis technique transforms equations from the time domain to the phasor domain.

2. Integro-differential equations get converted into linear equations with no sinusoidal functions.

3. After solving for the desired variable--such as a particular voltage or current-- in the phasor domain, conversion back to the time domain provides the same solution in steady state that would have been obtained had the original integro-differential equations been solved for steady state solution entirely in the time domain

Phasor Domain

$$v(t) = V_0 \cos(\omega t + \phi)$$
$$= \Re e[\underbrace{V_0 e^{j\phi}}_{\text{Phasor counterpart } v(t)} e^{j\omega t}]$$

$$\boxed{sin(\omega t) = \cos(\omega t - 90^0)}$$

$$\boxed{j^2 = -1 = 1\angle 180^0 = 1\angle -180^0}$$

Time Domain		**Phasor Domain**
$v(t) = V_0 \cos \omega t$	$\leftrightarrow$	$\mathbf{V} = V_0$
$v(t) = V_0 \cos(\omega t + \phi)$	$\leftrightarrow$	$\mathbf{V} = V_0 e^{j\phi} = V_0 \angle \phi$

Time and Phasor Domain

$x(t)$		$\mathbf{X}$
$A \cos \omega t$	$\leftrightarrow$	A
$A \cos(\omega t + \phi)$	$\leftrightarrow$	$Ae^{j\phi} = A\angle\phi$
$-A \cos(\omega t + \phi)$	$\leftrightarrow$	$Ae^{j(\phi \pm \pi)}$
$A \sin \omega t$	$\leftrightarrow$	$Ae^{-j\pi/2} = -jA$
$A \sin(\omega t + \phi)$	$\leftrightarrow$	$Ae^{j(\phi - \pi/2)}$
$-A \sin(\omega t + \phi)$	$\leftrightarrow$	$Ae^{j(\phi + \pi/2)}$
$\frac{d}{dt}(x(t))$	$\leftrightarrow$	$j\omega \mathbf{X}$
$\frac{d}{dt}[A \cos(\omega t + \phi)]$	$\leftrightarrow$	$j\omega A e^{j\phi}$
$\int x(t)\, dt$	$\leftrightarrow$	$\frac{1}{j\omega} \mathbf{X}$
$\int A \cos(\omega t + \phi)\, dt$	$\leftrightarrow$	$\frac{1}{j\omega} A e^{j\phi}$

It is much easier to deal with exponentials in the phasor domain than sinusoidal relations in the time domain.

You just need to track magnitude/phase, knowing that everything is at frequency ω.

Principles of Electrical Engineering I

Time Domain, Phasor Domain, and Impedance

Elements R, L, and C in time-domain and phasor-domain

Basics: Let us use $\cos(\omega t)$ as a reference time domain signal for a known frequency ω. Then the time domain signal $A\cos(\omega t + \theta)$ is characterized by its amplitude A and phase angle θ. As such the time domain signal $A\cos(\omega t + \theta)$ is transformed as a phasor $A\angle\theta$ in phasor domain and viceversa:

$$\text{Time domain: } A\cos(\omega t + \theta) \quad \Leftrightarrow \quad \text{phasor-domain: } A\angle\theta.$$

Resistance:

Time domain: $\quad v = Ri$

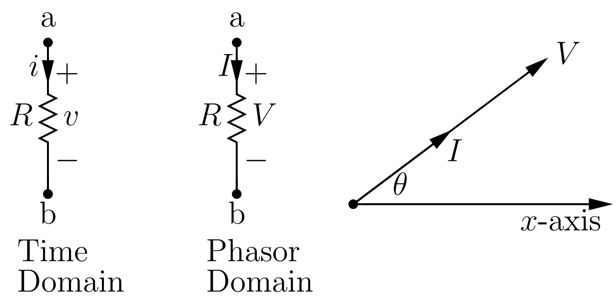
$$v(t) = V_m \cos(\omega t + \theta)$$

$$i(t) = \frac{V_m}{R}\cos(\omega t + \theta)$$

Phasor-domain:

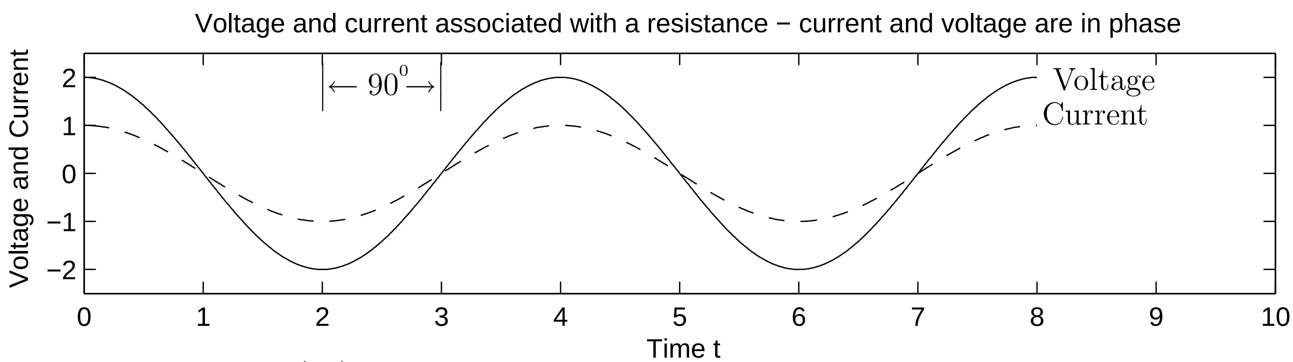
$$V = V_m\angle\theta \quad \text{and} \quad I = \frac{V_m}{R}\angle\theta$$

Therefore, $\boxed{V = RI}$ or $\boxed{\frac{V}{I} = R.}$



Impedance of resistance R is simply R.

The voltage and current are in phase.



We can use also $\sin(\omega t)$ as a reference time domain signal for a known frequency ω. Then the time domain signal $A\sin(\omega t + \theta)$ is characterized by its amplitude A and phase angle θ. As such the time domain signal $A\sin(\omega t + \theta)$ is transformed as a phasor $A\angle\theta$ in phasor domain and viceversa:

$$\text{Time domain: } A\sin(\omega t + \theta) \quad \Leftrightarrow \quad \text{phasor-domain: } A\angle\theta.$$

Time domain: $\quad v = Ri$

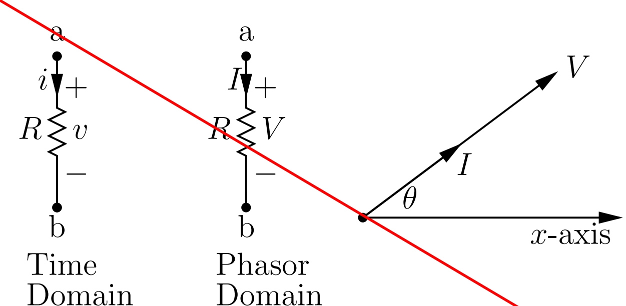
$$v(t) = V_m \sin(\omega t + \theta)$$

$$i(t) = \frac{V_m}{R}\sin(\omega t + \theta)$$

Phasor-domain:

$$V = V_m\angle\theta \quad \text{and} \quad I = \frac{V_m}{R}\angle\theta$$

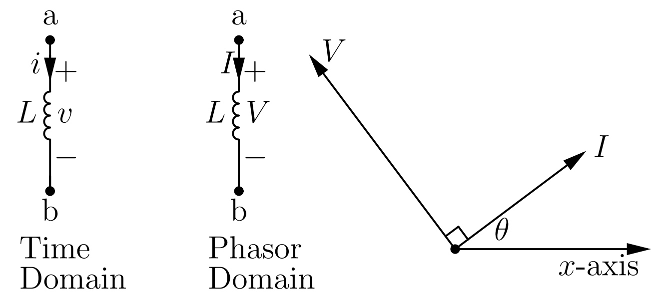
Therefore, $V = RI$ or $\frac{V}{I} = R.$



Again, Impedance of resistance R is simply R.

The voltage and current are in phase.

Inductance:



Time domain: $v = L\dfrac{di}{dt}$

$i(t) = I_m \cos(\omega t + \theta)$

$v(t) = L\dfrac{di}{dt} = -\omega L I_m \sin(\omega t + \theta)$

$\quad = \omega L I_m \cos(\omega t + \theta + 90^0)$

Phasor-domain:

$\boxed{j = 1\underline{/90^0}}$

$I = I_m\angle\theta$ and $V = \omega L I_m \underline{/\theta + 90^0}$

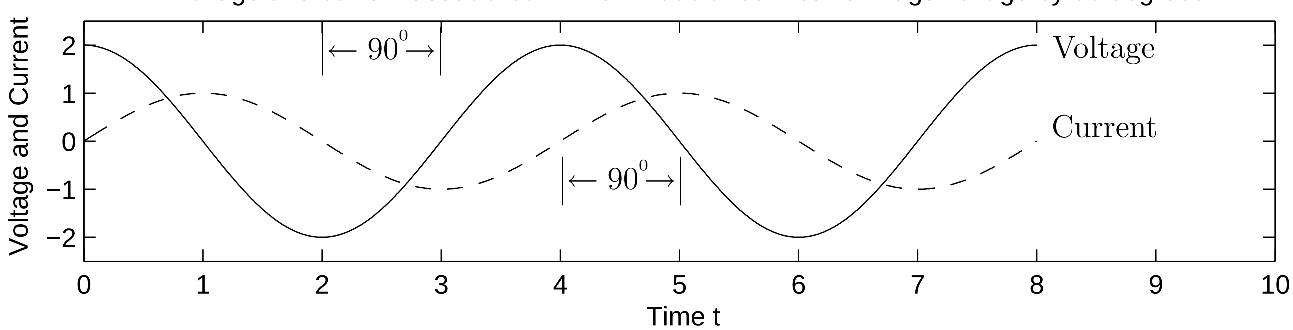
Therefore, $\dfrac{V}{I} = \omega L\underline{/90^0}$ or $\boxed{\dfrac{V}{I} = j\omega L.}$

$\boxed{V = j\omega L I}$

Impedance of inductance L is $j\omega L$.

The current lags the voltage by 90^0.

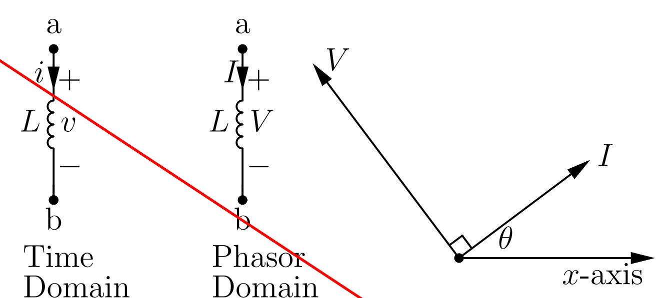
Voltage and current associated with an inductance − current lags voltage by 90 degrees



We note that the peak of voltage across inductance occurs first at some time point, then the peak of current through the inductance follows it after 90^0, that is after certain time-elapse such that $\omega\times$ time-elapse equals 90^0. This implies that **the current phasor lags the voltage phasor by 90^0**.

We can use also $\sin(\omega t)$ as a reference time domain signal for a known frequency ω. Then the time domain signal $A\sin(\omega t + \theta)$ is characterized by its amplitude A and phase angle θ. As such the time domain signal $A\sin(\omega t + \theta)$ is transformed as a phasor $A\angle\theta$ in phasor domain and viceversa:

$$\text{Time domain: } A\sin(\omega t + \theta) \quad \Leftrightarrow \quad \text{phasor-domain: } A\angle\theta.$$



Time domain: $v = L\dfrac{di}{dt}$

$i(t) = I_m \sin(\omega t + \theta)$

$v(t) = L\dfrac{di}{dt} = \omega L I_m \cos(\omega t + \theta)$

$\quad = \omega L I_m \sin(\omega t + \theta + 90^0)$

Phasor-domain:

$I = I_m\angle\theta$ and $V = \omega L I_m \underline{/\theta + 90^0}$

Therefore, $\dfrac{V}{I} = \omega L\underline{/90^0}$ or $\dfrac{V}{I} = j\omega L.$

Again, Impedance of inductance L is $j\omega L$.

The current lags the voltage by 90^0.

Capacitance:

Time domain: $i = C\dfrac{dv}{dt}$

$v(t) = V_m \cos(\omega t + \theta)$

$i(t) = C\dfrac{dv}{dt} = -\omega C V_m \sin(\omega t + \theta)$

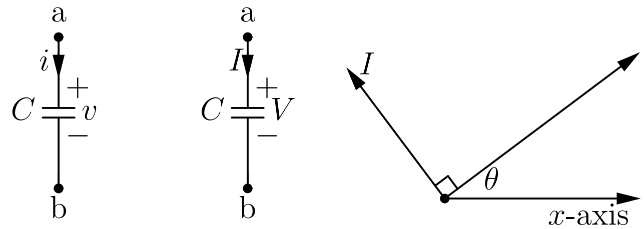
$\quad = \omega C V_m \cos(\omega t + \theta + 90^0)$

Phasor-domain:

$V = V_m \angle\theta$ and $I = \omega C V_m \angle \theta + 90^0$

$I = j\omega C V$

Therefore, $\dfrac{V}{I} = \dfrac{1}{\omega C \angle 90^0}$ or $\boxed{\dfrac{V}{I} = \dfrac{1}{j\omega C}}$.



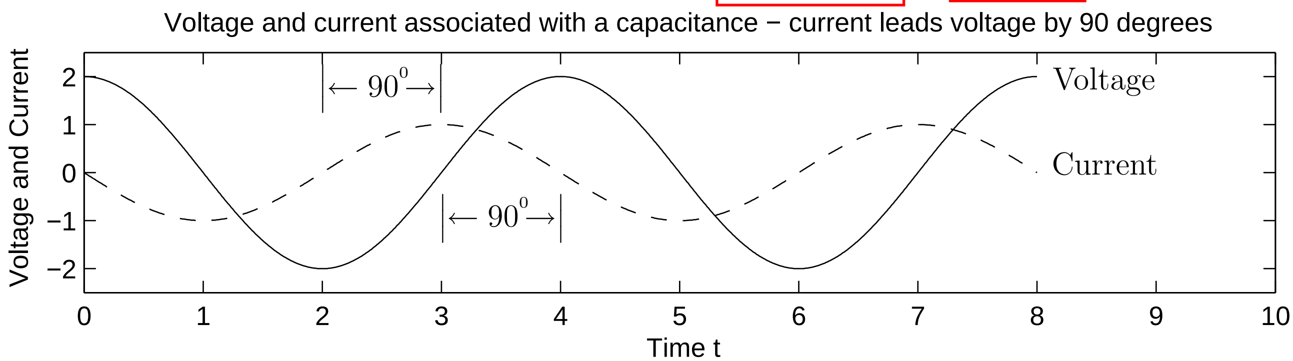
Time Domain

Phasor Domain

Impedance of capacitance C is $\frac{1}{j\omega C}$.

The current leads the voltage by 90^0.

$\boxed{\dfrac{1}{j\omega C} = \dfrac{-j}{\omega C}}$ $\boxed{\dfrac{1}{j} = -j}$



We note that the peak of current through capacitance occurs first at some time point, then the peak of voltage across the capacitance follows it after 90^0, that is after certain time-elapse such that $\omega\times$ time-elapse equals 90^0. This implies that the **current phasor leads the voltage phasor by 90^0**.

We can use also $\sin(\omega t)$ as a reference time domain signal for a known frequency ω. Then the time domain signal $A\sin(\omega t + \theta)$ is characterized by its amplitude A and phase angle θ. As such the time domain signal $A\sin(\omega t + \theta)$ is transformed as a phasor $A\angle\theta$ in phasor domain and viceversa:

$$\text{Time domain: } A\sin(\omega t + \theta) \quad \Leftrightarrow \quad \text{phasor-domain: } A\angle\theta.$$

Time domain: $i = C\dfrac{dv}{dt}$

$v(t) = V_m \sin(\omega t + \theta)$

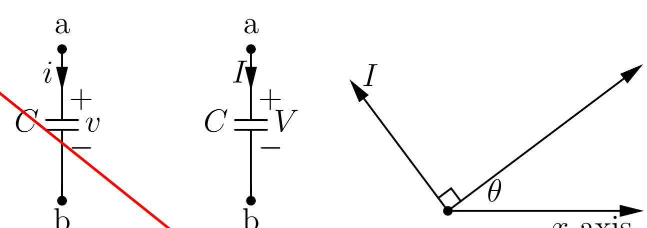
$i(t) = C\dfrac{dv}{dt} = \omega C V_m \cos(\omega t + \theta)$

$\quad = \omega C V_m \sin(\omega t + \theta + 90^0)$

Phasor-domain:

$V = V_m \angle\theta$ and $I = \omega C V_m \angle \theta + 90^0$

Therefore, $\dfrac{V}{I} = \dfrac{1}{\omega C \angle 90^0}$ or $\dfrac{V}{I} = \dfrac{1}{j\omega C}$.



Time Domain

Phasor Domain

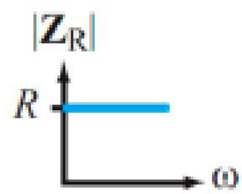
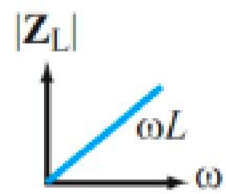
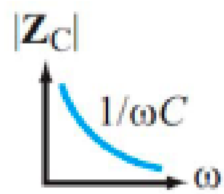
Again, Impedance of capacitance C is $\frac{1}{j\omega C}$.

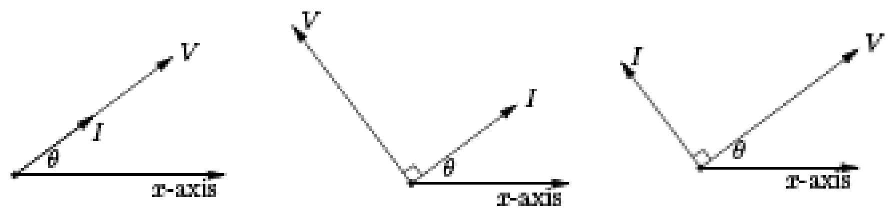
The current leads the voltage by 90^0.

A very important relationship in phasor domain

Summary of *R*, *L*, *C*

$$\frac{1}{j\omega C} = \frac{-j}{\omega C}$$

Property	R	L	C
v–i	$v = Ri$	$v = L\,\dfrac{di}{dt}$	$i = C\,\dfrac{dv}{dt}$
V–I	$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$
Z = Impedance	R	$j\omega L$	$\dfrac{1}{j\omega C}$
dc equivalent	R	Short circuit	Open circuit
High-frequency equivalent	R	Open circuit	Short circuit
Frequency response			



Voltage across and current through a resistance are in phase

Current through an inductance lags the voltage across it by 90 degrees

Current through a capacitance leads the voltage across it by 90 degrees

Phasor domain is also called Frequency domain.

Signal tranformation:

$$\begin{array}{ccc} \textit{Time domain} & & \textit{Phasor domain} \\ A\cos(\omega t + \theta) & \Longleftrightarrow & Ae^{j\theta} = A\underline{/\ \theta}. \end{array}$$

Principles of Electrical Engineering I

Terminology

The ratio of voltage to current phasor of a particular branch is called the **Impedance** of that branch. The unit of impedance is **ohm**. Similarly, the ratio of current to voltage phasor of a particular branch is called the **Admittance** of that branch. The unit of admittance is **siemen**, however it is also often called **mho**.

$$\text{Impedance} = \frac{\text{Voltage Phasor}}{\text{Current Phasor}} = Z = \boxed{R} + j\,\boxed{X}.$$

R= Real Part of Impedance = **Resistance**.
The unit of Resistance is also **ohm**.
X= Imaginary Part of Impedance = **Reactance**.
The unit of Reactance is also **ohm**.

$$\text{Admittance} = \frac{\text{Current Phasor}}{\text{Voltage Phasor}} = Y = \boxed{G} + j\,\boxed{B}.$$

G= Real Part of Admittance = **Conductance**.
The unit of Conductance is also **mho**.
B= Imaginary Part of Admittance =**Susceptance**.
The unit of Susceptance is also **mho**.

Resistor	$\mathbf{Z} = R$	$\mathbf{Y} = 1/R$
Inductor	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = 1/j\omega L$
Capacitor	$\mathbf{Z} = 1/j\omega C$	$\mathbf{Y} = j\omega C$

$$\boxed{\frac{1}{j} = -j} \qquad \boxed{\frac{1}{j\omega C} = \frac{-j}{\omega C}}$$

We note that

$$Y = G + j\,B = \frac{1}{Z} = \frac{1}{R + j\,X} = \frac{R - j\,X}{(R + j\,X)(R - j\,X)} = \frac{R - j\,X}{R^2 + X^2}.$$

$$G = \frac{R}{R^2 + X^2} \quad \text{and} \quad B = \frac{-X}{R^2 + X^2}.$$

$$Z = R + j\,X = \frac{1}{Y} = \frac{1}{G + j\,B} = \frac{G - j\,B}{(G + j\,B)(G - j\,B)} = \frac{G - j\,B}{G^2 + B^2}.$$

$$R = \frac{G}{G^2 + B^2} \quad \text{and} \quad X = \frac{-B}{G^2 + B^2}.$$

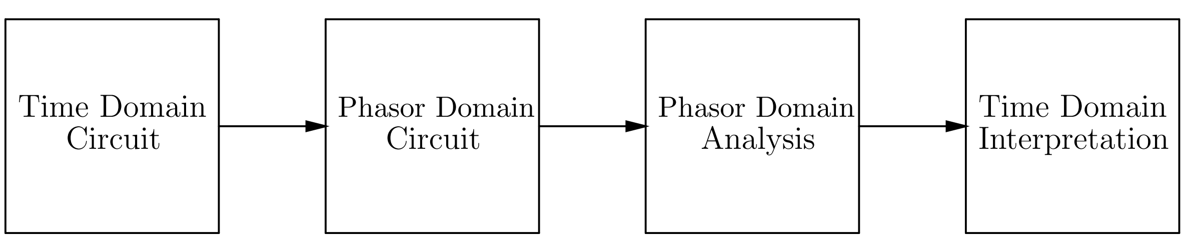
We emphasize that in general,

$$R \neq \frac{1}{G}\,,\ G \neq \frac{1}{R}\,,\ B \neq \frac{-1}{X} \ \text{ and } \ X \neq \frac{-1}{B}.$$

Philosophy of Sinusoidal Steady State Analysis:

As depicted in the following block diagram, Sinusoidal Steady State Analysis involves four steps,

- Time domain circuit is given.
- Time domain circuit is transformed to its equivalent phasor domain circuit.
- All the analysis is done in phasor domain.
- Phasor domain analysis is interpreted in time domain.



Sinusoidal Steady State Analysis

Signal tranformation:

$$\boxed{\begin{array}{lcl} \mathit{Time\ domain} & & \mathit{Phasor\ domain} \\ A\cos(\omega t + \theta) & \Longleftrightarrow & Ae^{j\theta} = A\underline{/\ \theta}. \end{array}}$$

Phasor domain is also called Frequency domain.

$$\text{Impedance} = \frac{\text{Voltage Phasor}}{\text{Current Phasor}}$$

Resistor	$\mathbf{Z} = R$	$\mathbf{Y} = 1/R$
Inductor	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = 1/j\omega L$
Capacitor	$\mathbf{Z} = 1/j\omega C$	$\mathbf{Y} = j\omega C$

Philosophy of Sinusoidal Steady State Analysis:

As depicted in the following block diagram, Sinusoidal Steady State Analysis involves four steps,

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Involves Complex Algebra

Time Domain Circuit → Phasor Domain Circuit → Phasor Domain Analysis → Time Domain Interpretation

Phasor Domain Analysis

Analysis in phasor domain (otherwise called frequency domain) is exactly the same as in the case of analysis with resistances alone.

We visualize now that each element has an impedance rather than just resistance.

Time domain signals are treated as phasors.

The algebra utilizes complex numbers rather than real numbers alone.

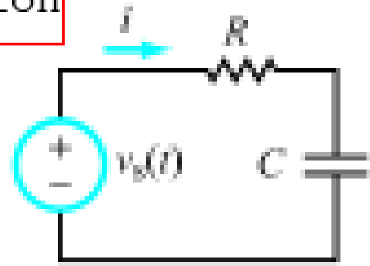
All the following aspects we developed earlier with resistances alone carry over when we use impedances. Conceptually, it is straight forward. The only complication is that we need to work with complex algebra.

- Series Parallel Combinations
- Voltage and Current division
- Δ-Y transformations
- Superposition method
- Source transformations
- Node Voltage Method
- Mesh Current Method
- Thevenin and Norton Equivalents

Example of a RC Circuit:

Step by Step Illustration

Determine the current $i(t)$ in steady state



$v_s(t) = 12\sin(\omega t - 45^\circ)$ (V)

$\omega = 1000 \text{ rad/sec}$

$R = \sqrt{3}\,\text{k}\Omega,\ C = 1\,\mu\text{F}.$

$\sin(\omega t) = \cos(\omega t - 90^\circ)$

$12\sin(\omega t - 45^\circ) = 12\cos(\omega t - 135^\circ)$

Step 1
Adopt Cosine Reference
(Time Domain)

Step 2
Transfer to Phasor Domain

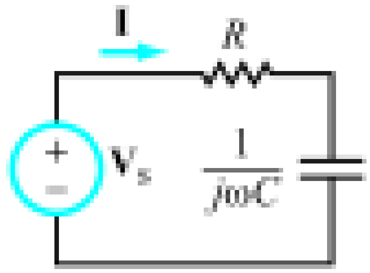
$i \rightarrow \mathbf{I}$

$v \rightarrow \mathbf{V}$

$R \rightarrow \mathbf{Z}_R = R$

$L \rightarrow \mathbf{Z}_L = j\omega L$

$C \rightarrow \mathbf{Z}_C = 1/j\omega C$



$\mathbf{V}_s = 12e^{-j135^\circ}$ (V)

$e^{j\theta} = 1\angle\theta$

$V_s = 12e^{-j135^\circ} = 12\angle -135^\circ$

Step 3
Cast Equations in
Phasor Form

$$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$$

$$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = 12e^{-j135^\circ}.$$

$$\mathbf{I} = \frac{12e^{-j135^\circ}}{R + \frac{1}{j\omega C}} = \frac{j12\omega C e^{-j135^\circ}}{1 + j\omega RC}.$$

Step 4
Solve for Unknown Variable
(Phasor Domain)

$$\mathbf{I} = \frac{\mathbf{V}_s}{R + \frac{1}{j\omega C}}$$

$R = \sqrt{3}\,\text{k}\Omega,\ C = 1\,\mu\text{F}.$

$\omega = 1000 \text{ rad/sec}$

$$\mathbf{I} = \frac{j12 \times 10^3 \times 10^{-6} e^{-j135^\circ}}{1 + j10^3 \times \sqrt{3} \times 10^3 \times 10^{-6}}$$

$$= \frac{j12e^{-j135^\circ}}{1 + j\sqrt{3}} \text{ mA.}$$

$$= \frac{12e^{-j135^\circ} \cdot e^{j90^\circ}}{2e^{j60^\circ}}$$

$$= 6e^{j(-135^\circ + 90^\circ - 60^\circ)}$$

$$= 6e^{-j105^\circ} \text{ mA.}$$

Step 5
Transform Solution
Back to Time Domain

$$i(t) = \Re e[\mathbf{I}e^{j\omega t}] = 6\cos(\omega t - 105^\circ) \text{ (mA)}$$

$$1 + j\sqrt{3} = \sqrt{1+3}\,e^{j\phi} = 2e^{j\phi},$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60^\circ.$$

$$(a + jb)(c + jd) = ac + jad + jbc + j^2bd = ac - bd + j(ad + bc)$$
$$(a + jb)(a - jb) = a^2 + jab - jab + b^2 = a^2 + b^2$$

Example: Determine the current $i(t)$ and voltage $v_c(t)$ in steady state when the input $v_g(t) = 30\sqrt{2}\cos(5000\,t)$ V.

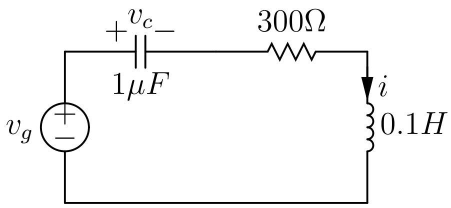


Figure 1: Time-domain circuit

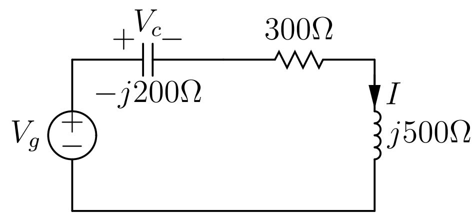


Figure 2: Phasor-domain circuit

Solution: Our method of analysis is to convert the given time domain circuit to a phasor domain circuit, do the required analysis in phasor domain, and then interpret the results of such an analysis in time domain. **Note that all circuit analysis concepts as well as algebra are associated with phasor domain.** Various steps of doing so are as follows:

- We transform the given time domain circuit to phasor domain as shown in Figure 2. To do so, we mark the impedance of each element, and replace the time domain input signal $v_g(t)$ by its phasor V_g. Note that by using $\cos(5000\,t)$ as the basis of phasor transformation, we see that

$$\underbrace{v_g(t) = 30\sqrt{2}\cos(5000\,t)}_{\text{Time-domain}} \quad \Rightarrow \quad \underbrace{V_g = 30\sqrt{2}\,\underline{/\,0^0}}_{\text{Phasor-domain}}.$$

- We observe that the input signal $v_g(t)$ has the angular frequency $\omega = 5000$ rad/sec.

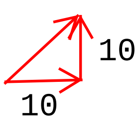
- Impedance of resistance R is R itself. For this circuit $R = 300\Omega$.

- Impedance of inductance L is $j\omega L\Omega$. For this circuit $j\omega L = j(5000)(0.1) = j500\Omega$.

- Impedance of capacitance C is $\frac{1}{j\omega C}\Omega$. For this circuit $\frac{1}{j\omega C} = \frac{1}{j(5000)(10^{-6})} = -j200\Omega$.

- We observe that all the impedances are in series and hence the total impedance Z seen by V_g equals $300 + j500 - j200 = 300 + j300$.

- Hence the phasor current I is given by



$$I = \frac{V_g}{Z} = \frac{30\sqrt{2}}{300 + j300} = \frac{\sqrt{2}}{10 + j10} = \frac{\sqrt{2}}{10\sqrt{2}\,\underline{/\,45^0}} = 0.1\,\underline{/\,-45^0}.$$

The above algebra could have been done slightly differently as

$$I = \frac{V_g}{Z} = \frac{30\sqrt{2}}{300 + j300} = \frac{\sqrt{2}}{10 + j10} = \frac{\sqrt{2}(1 - j1)}{(10)(1 + j1)(1 - j1)} = \frac{\sqrt{2}(1 - j1)}{20} = \frac{0.1}{\sqrt{2}}(1 - j1) = 0.1\,\underline{/\,-45^0}.$$

- The voltage $V_c = I(-j200) = (0.1\,\underline{/\,-45^0})(-j200) = (20\,\underline{/\,-45^0})(1\,\underline{/\,-90^0}) = 20\,\underline{/\,-135^0}$.

- We now can go back to time domain by transforming each phasor into its corresponding time function,

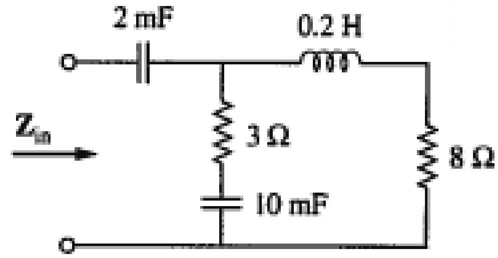
$$\underbrace{I = 0.1\,\underline{/\,-45^0}}_{\text{Phasor-domain}} \quad \Rightarrow \quad \underbrace{i(t) = 0.1\cos(5000\,t - 45^0)}_{\text{Time-domain}},$$

and

$$\underbrace{V_c = 20\,\underline{/\,-135^0}}_{\text{Phasor-domain}} \quad \Rightarrow \quad \underbrace{v_c(t) = 20\cos(5000\,t - 135^0)}_{\text{Time-domain}}.$$

Example: Find the input impedance of the circuit

$$\omega = 50 \text{ rad/sec}$$



Solution:

Let

Z_1 = Impedance of the 2-mF capacitor

Z_2 = Impedance of the 3-Ω resistor in series with the 10-mF capacitor

Z_3 = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

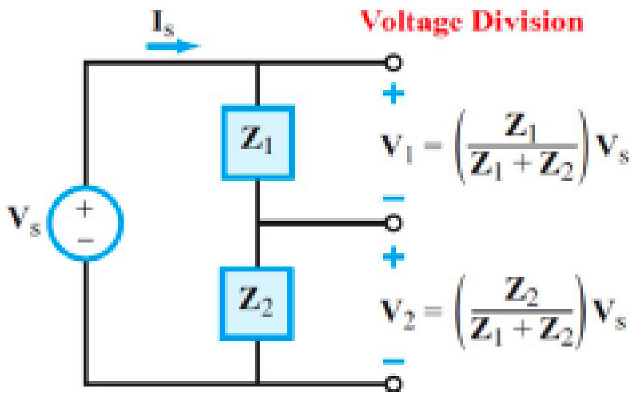
$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \| \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

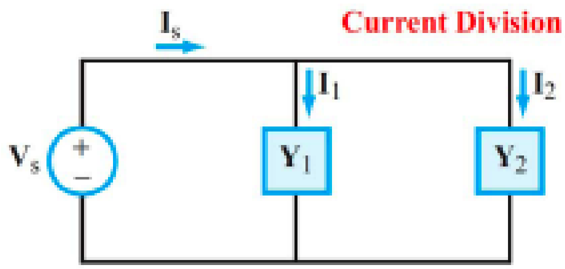
$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus,

$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$

Voltage & Current Division

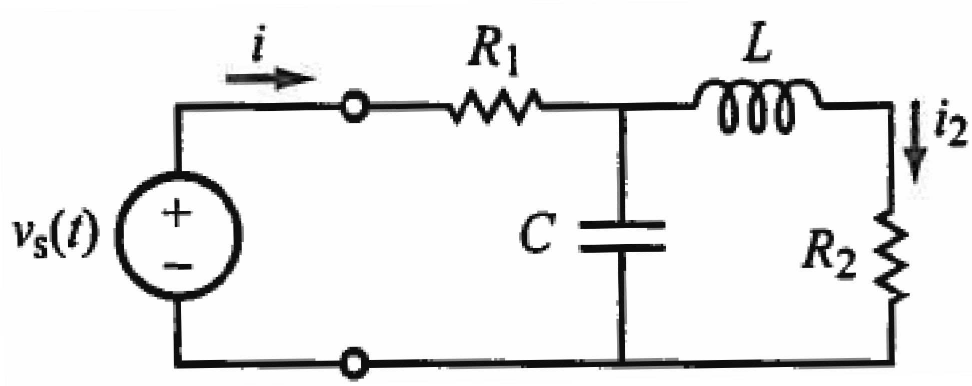




$$I_1 = \left(\frac{Y_1}{Y_1 + Y_2}\right) I_s = \left(\frac{Z_2}{Z_1 + Z_2}\right) I_s$$

$$I_2 = \left(\frac{Y_2}{Y_1 + Y_2}\right) I_s = \left(\frac{Z_1}{Z_1 + Z_2}\right) I_s$$

Example: Determine the current i_2 through R_2 if $R_1 = 10\Omega$, $R_2 = 30\Omega$, $L = 2\ \mu\text{H}$, and $C = 10$ nF. Note carefully the notations μ and n.



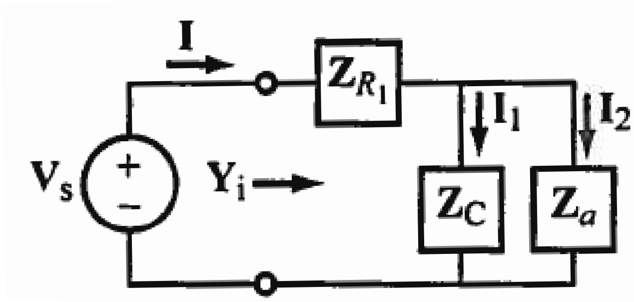
$$v_s(t) = 4\cos(10^7 t - 75^\circ)\ \text{V}$$

$$\mathbf{V}_s = 4e^{-j75^\circ}\ \text{V}$$

$$\mathbf{Z}_{R_1} = R_1 = 10\ \Omega,$$

$$\mathbf{Z}_C = \frac{-j}{\omega C} = \frac{-j}{10^7 \times 10^{-8}} = -j10\ \Omega$$

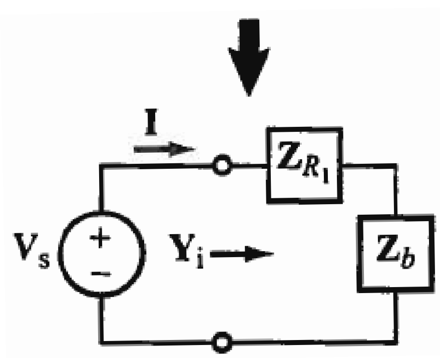
$$\mathbf{Z}_a = R_2 + j\omega L = 30 + j20\ \Omega$$



$$\mathbf{Z}_a = R_2 + j\omega L$$

$$\mathbf{Z}_C = -\frac{j}{\omega C}$$

Phasor domain



$$\mathbf{Z}_b = \mathbf{Z}_C \parallel \mathbf{Z}_a$$

$$\mathbf{Z}_b = \mathbf{Z}_C \parallel \mathbf{Z}_a$$

$$= \frac{(-j10)(30 + j20)}{-j10 + 30 + j20} = \frac{20 - j30}{3 + j1}$$

$$= \frac{(20 - j30)}{(3 + j1)} \frac{(3 - j1)}{(3 - j1)} = (3 - j11)\ \Omega.$$

The input impedance is

$$\mathbf{Z}_i = \mathbf{Z}_{R_1} + \mathbf{Z}_b = 10 + 3 - j11 = (13 - j11)\ \Omega,$$

and its reciprocal is

Not used

$$\mathbf{Y}_i = \frac{1}{\mathbf{Z}_i}$$
$$= \frac{1}{13 - j11} \times \frac{13 + j11}{13 + j11}$$
$$= \frac{13 + j11}{169 + 121} = (4.5 + j3.8) \times 10^{-2}\ \text{S}.$$

(b) The current **I** is given by

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}_i} = \frac{4e^{-j75^\circ}}{13 - j11} = \frac{4e^{-j75^\circ}}{17.03e^{-j40.2^\circ}} = 0.235e^{-j34.8^\circ}\ \text{A}.$$

By current division

$$\mathbf{I}_2 = \frac{\mathbf{Z}_C}{\mathbf{Z}_a + \mathbf{Z}_C}\,\mathbf{I}$$
$$= \frac{-j10}{30 + j20 - j10} \times 0.235e^{-j34.8^\circ}$$
$$= \frac{2.35e^{-j34.8^\circ} \cdot e^{-j90^\circ}}{31.6e^{j18.4^\circ}} = 7.4 \times 10^{-2} e^{-j143.2^\circ}\ \text{A}.$$

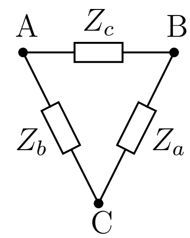
$$i_2(t) = 7.4 \times 10^{-2} \cos(10^7 t - 143.2^\circ)\ \text{A}.$$

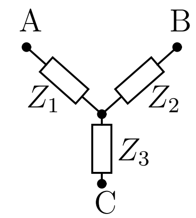
$\Delta-$ Y equivalents

$$Z_1 = \frac{Z_b Z_c}{Z_a+Z_b+Z_c} \qquad Z_a = \frac{Z_1Z_2+Z_2Z_3+Z_3Z_1}{Z_1}$$

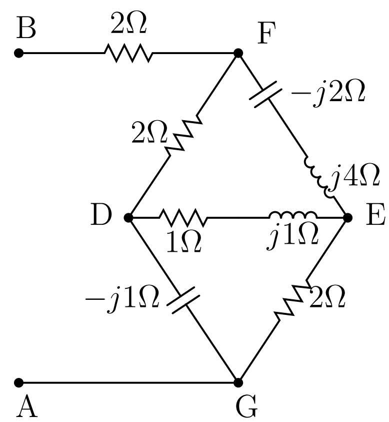
$$Z_2 = \frac{Z_a Z_c}{Z_a+Z_b+Z_c} \qquad Z_b = \frac{Z_1Z_2+Z_2Z_3+Z_3Z_1}{Z_2}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a+Z_b+Z_c} \qquad Z_c = \frac{Z_1Z_2+Z_2Z_3+Z_3Z_1}{Z_3}$$





Example: Consider the bridge circuit shown and determine the impedance seen at the terminals A and B. Although concepts are simple, we will use this example to illustrate complex algebra.
In this example, we need to use $\Delta-$ Y equivalents. There are many possible ways of doing so. We will transform the Δ formed by the nodes D, E, and F into a Y. The relevant computations are as shown below:



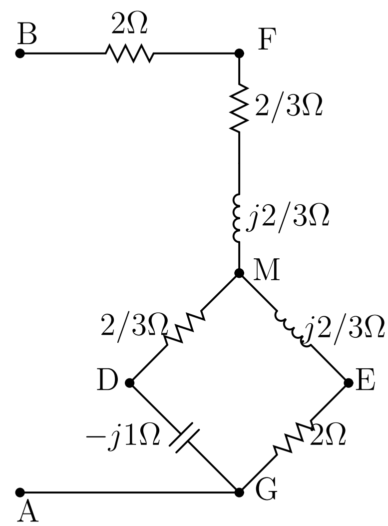
$$\frac{(2)(-j2+j4)}{2-j2+j4+1+j1} = \frac{(2)(j2)}{3+j3} = \frac{(j4)(3-j3)}{(3+j3)(3-j3)}$$
$$= \frac{j12-j^2 12}{9+9} = \frac{12+j12}{18} = \frac{2}{3}(1+j1) = \frac{2\sqrt{2}}{3}\angle 45^0\,\Omega$$

Alternatively, we could have computed some of the above calculations in polar form as shown below:

$$\frac{(2)(-j2+j4)}{2-j2+j4+1+j1} = \frac{(2)(j2)}{3+j3} = \frac{4\angle 90^0}{\sqrt{18}\angle 45^0}$$
$$= \frac{4\angle 45^0}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}\angle 45^0 = \frac{2}{3}(1+j1)\Omega.$$

We computed above one leg of Y. The other two legs are easy to compute in a similar way,

$$\frac{(2)(1+j1)}{3+j3} = \frac{2}{3}\Omega, \qquad \frac{(1+j1)(-j2+j4)}{3+j3} = \frac{(1+j1)(j2)}{3+j3} = \frac{j2}{3}\Omega.$$



The simplified circuit after $\Delta-$ Y transformation is shown above on the right. We recognize two parallel paths between the nodes G and M. We can simplify this into one path as

$$\frac{(2/3-j1)(2+j2/3)}{2/3-j1+2+j2/3} = \frac{(2-j3)(6+j2)}{6-j9+18+j6} = \frac{12-j18+j4-j^2 6}{24-j3}$$
$$= \frac{18-j14}{24-j3} = \frac{(18-j14)(24+j3)}{24^2+3^2} = \frac{474-j282}{585} = 0.81-j0.482\,\Omega.$$

We can now determine the impedance seen at the terminals A and B as

$$Z = 2+\frac{2}{3}+\frac{j2}{3}+0.81-j0.482 = 3.477+j0.1846\,\Omega.$$

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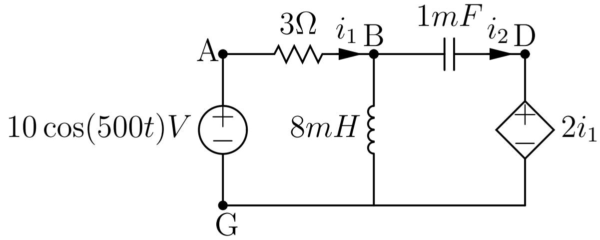
Quiz 5

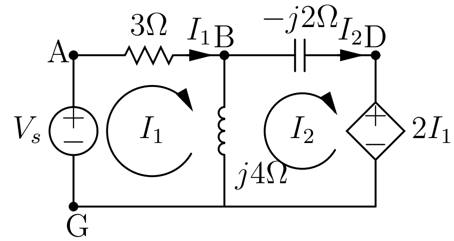
Student's name in capital letters:

Assume that the circuit given is in steady state. Construct a phasor domain equivalent circuit. Determine phasor values of I_1 and I_2. Then, determine the time signals i_1 and i_2.



The radian frequency ω of the input signal is 500 rad/sec.
The impedance of the inductance is $j\omega L = j500(8m) = j4\Omega$.
The impedance of the capacitance is $\frac{1}{j\omega C} = \frac{1}{j500(1m)} = -j2\Omega$.

The phasor domain equivalent circuit is shown on the right along with two mesh currents I_1 and I_2. The input phasor $V_s = 10\angle 0^{\circ}$.



We can write the two mesh equations as

$$3I_1 + j4(I_1 - I_2) = 10\angle 0^{\circ} \quad \Rightarrow \quad (3 + j4)I_1 - j4I_2 = 10,$$

$$j4(I_2 - I_1) - j2I_2 + 2I_1 = 0 \quad \Rightarrow \quad (2 - j4)I_1 + j2I_2 = 0.$$

By multiplying the second equation by 2 and adding the resulting equation to the first equation, we get

$$(7 - j4)I_1 = 10 \quad \Rightarrow \quad I_1 = \frac{10}{7 - j4} = \frac{10(7 + j4)}{49 + 16} = 1.077 + j0.6154 = 1.24\angle 29.75^{\circ} A.$$

Substituting for I_1 in one of the above equations and simplifying, we get

$$I_2 = 2.77\angle 56.3^{\circ} A.$$

Now going back to time-domain, we get

$$i_1 = 1.24\cos(500t + 29.75^{\circ})A \quad \text{and} \quad i_2 = 2.77\cos(500t + 56.3^{\circ})A.$$

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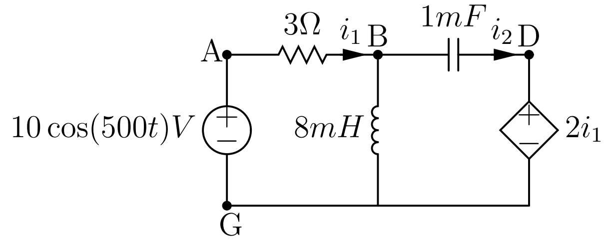
Quiz 5

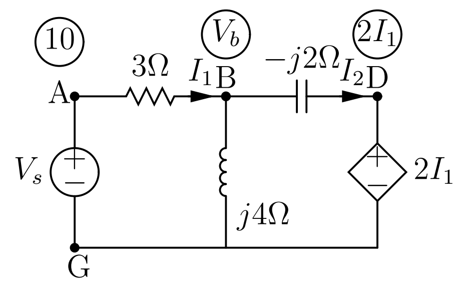
Student's name in capital letters:

Assume that the circuit given is in steady state. Construct a phasor domain equivalent circuit. Determine phasor values of I_1 and V_b, the phasor voltage of node B with respect to G. Then, determine the time functions i_1 and v_b.



The radian frequency ω of the input signal is 500 rad/sec.
The impedance of the inductance is $j\omega L = j500(8m) = j4\Omega$.
The impedance of the capacitance is $\frac{1}{j\omega C} = \frac{1}{j500(1m)} = -j2\Omega$.

The phasor domain equivalent circuit is shown on the right along with the node voltages marked with respect to G. The input phasor $V_s = 10\underline{/\ 0^0}$.



We can easily determine I_1 as

$$I_1 = \frac{10 - V_b}{3}.$$

We can now write the node equation at point B as

$$\frac{V_b - 10}{3} + \frac{V_b}{j4} + \frac{V_b - 2\frac{10-V_b}{3}}{-j2} = 0 \quad \Rightarrow \quad \frac{V_b - 10}{3} + \frac{V_b}{j4} + \frac{5V_b - 20}{-j6} = 0.$$

After multiplying with $j12$ throughout, the above equation simplifies to

$$(V_b - 10)j4 + 3V_b + (5V_b - 20)(-2) = 0 \quad \Rightarrow \quad (-7 + j4)V_b = -40 + j40.$$

This implies that

$$V_b = \frac{-40 + j40}{-7 + j4} = \frac{(-40 + j40)(-7 - j4)}{49 + 16} = \frac{440 - j120}{65} = 6.7692 - j1.8461 = 7.0164\underline{/\ -15.}255^0 V.$$

Having determined V_b, we can now determine

$$I_1 = \frac{10 - V_b}{3} = I_1 = \frac{10 - 6.7692 + j1.8461}{3} = \frac{3.2308 + j1.8461}{3} = 1.077 + j0.6154 = 1.24\underline{/\ 29}.75^0 A.$$

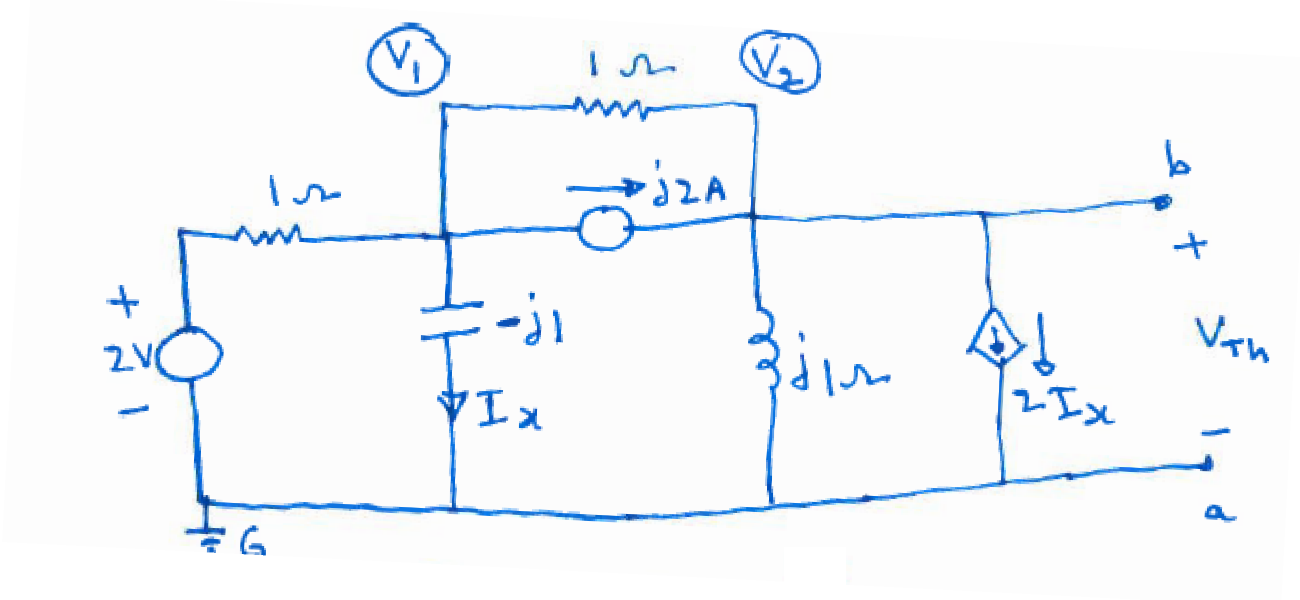
We can now translate the phasors I_1 and V_b into time domain as

$$i_1 = 1.24\cos(500t + 29.75^0)A \quad \text{and} \quad v_b = 7.0164\cos(500t - 15.255^0)V.$$

Example:

Determine the open circuit voltage V_{Th} across the terminals a and b in the circuit shown. Use any method you feel is convenient. Note that one method might be much easier than the other. So think carefully which method is better suited to the given problem.

For numerical simplicity in solving the equations, we inform you that $I_x = 0.4(2-j)$ A. This information should not be used to set up the equations.



Node voltage method is better suited than the mesh current method. Node voltages are marked with respect to the ground G. We note that $V_2 = V_{Th}$.

We note that $I_x = \frac{V_1}{-j} = jV_1$.

$$\frac{1}{j} = -j$$

Node equation at V_1 is

$$V_1 - 2 + jV_1 + V_1 - V_2 + j2 = 0 \quad \Rightarrow \quad V_1(2+j) - V_2 = 2(1-j).$$

Node equation at V_2 is

$$-jV_2 + j2V_1 + V_2 - V_1 - j2 = 0 \quad \Rightarrow \quad V_1(-1+j2) + V_2(1-j) = j2.$$

Using the hint given, we note that $V_1 = \frac{I_x}{j} = -j0.4(2-j) = -0.4(1+j2)$.

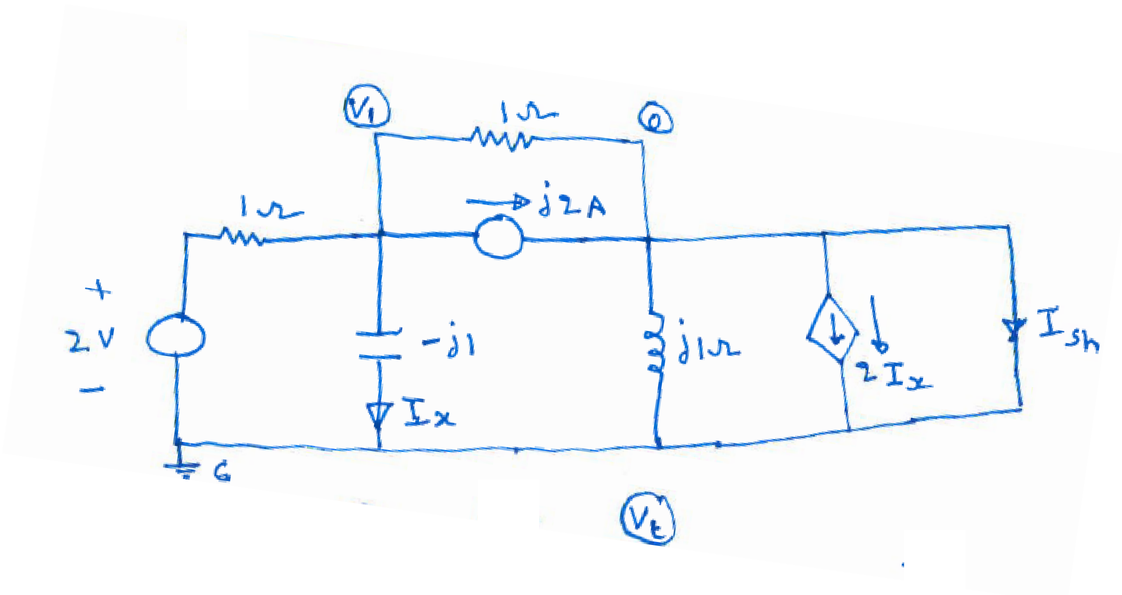
By utilizing the equation at V_1, we get

$$\begin{aligned} V_2 &= V_1(2+j) - 2(1-j) = -0.4(1+j2)(2+j) - 2(1-j) \\ &= -0.4(2+j+j4-2) - 2 + j2 = -j2 - 2 + j2 = -2. \end{aligned}$$

Thus

$$V_{Th} = V_2 = -2 \ V.$$

Determine the short circuit current I_{sh} through the terminals a and b in the circuit shown. Use any method you feel is convenient. Note that one method might be much easier than the other. So think carefully which method is better suited to the given problem.



Node voltage method is better suited than the mesh current method. Node voltages are marked with respect to the ground G.

We note that $I_x = \frac{V_1}{-j} = jV_1$.

$$\frac{1}{j} = -j$$

Node equation at V_1 is

$$V_1 - 2 + jV_1 + V_1 + j2 = 0 \quad \Rightarrow \quad V_1(2+j) = 2(1-j) \quad \Rightarrow \quad V_1 = \frac{2(1-j)}{2+j}$$

$$\Rightarrow \quad V_1 = \frac{2}{5}(1-j)(2-j) = \frac{2}{5}(2 - j2 - j - 1) = \frac{2}{5}(1 - j3).$$

Another node equation is

$$j2V_1 - V_1 - j2 + I_{sh} = 0 \quad \Rightarrow \quad I_{sh} = V_1(1 - j2) + j2.$$

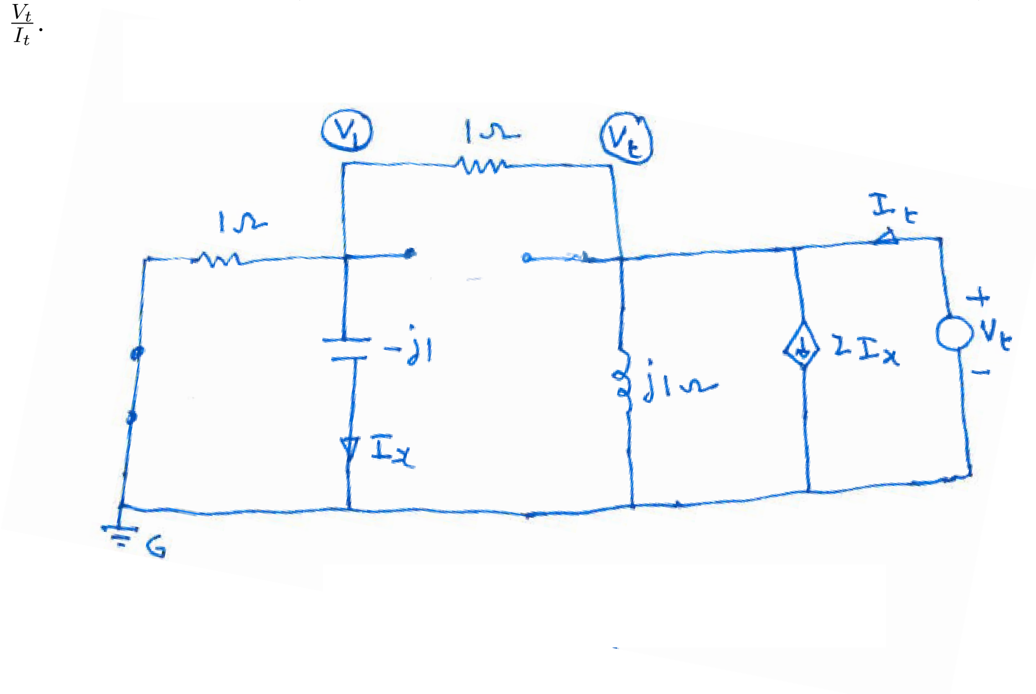
Substituting for $V_1 = \frac{2}{5}(1 - j3)$ and simplifying, we get

$$I_{sh} = V_1(1-j2) + j2 = \frac{2}{5}(1-j3)(1-j2) + j2 = \frac{2}{5}(1 - j3 - j2 - 6) + j2 = -2\ A.$$

At first, determine the Thevenin impedance $Z_{Th} = \frac{V_{Th}}{I_{sh}}$ where V_{Th} and I_{sh} are as determined in the previous pages.

$$Z_{Th} = \frac{V_{Th}}{I_{sh}} = \frac{-2}{-2} = 1.$$

Another way of determining Thevenin impedance Z_{Th} is to determine the impedance as seen from the terminals a and b of the given circuit in which all independent sources are set to zero. Such a life-less circuit is shown below. In order to determine the impedance as seen from the terminals a and b, an external source V_t is connected across a and b, determine the ratio $\frac{V_t}{I_t}$.



Node voltage method is better suited than the mesh current method. Node voltages are marked with respect to the ground G.

We note that $I_x = \frac{V_1}{-j} = jV_1$.

$$\boxed{\frac{1}{j} = -j}$$

Node equation at V_1 is

$$V_1 + jV_1 + V_1 - V_t = 0 \quad \Rightarrow \quad V_1(2+j) - V_t = 0 \quad \Rightarrow \quad V_1 = V_t \frac{1}{2+j} = \frac{V_t}{5}(2-j).$$

Another node equation is

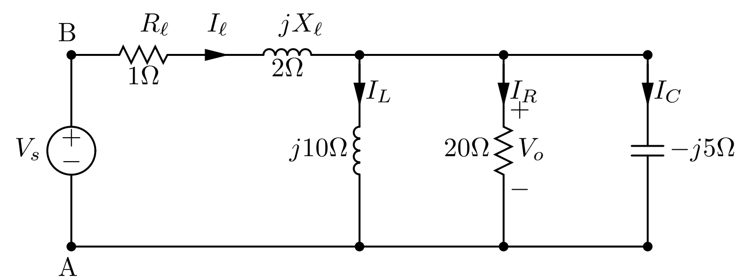
$$\begin{aligned} I_t = 2I_x - jV_t + V_t - V_1 \quad &\Rightarrow \quad I_t = j2V_1 - jV_t + V_t - V_1 = (-1+j2)V_1 + (1-j)V_t \\ &\Rightarrow \quad I_t = (-1+j2)\frac{V_t}{5}(2-j) + (1-j)V_t \\ &\Rightarrow \quad I_t = \frac{V_t}{5}(-2+j+j4+2) + (1-j)V_t = V_t. \end{aligned}$$

Thus

$$Z_{Th} = \frac{V_t}{I_t} = 1.$$

Example

A typical power distribution circuit is shown on the right. The required output voltage is $V_o = 110$ V. Determine the supply voltage V_s and line current I_ℓ. Also, draw the phasor diagram of all the variables.



We will consider the reference phasor as the output voltage $V_o = 110$. It is then straight forward to determine the following currents,

$$I_R = \frac{110}{20} = 5.5\,A, \quad I_L = \frac{110}{j10} = -j11\,A, \quad I_C = \frac{110}{-j5} = j22\,A.$$

The line current I_ℓ is then given by

$$I_\ell = I_R + I_L + I_C = 5.5 - j11 + j22 = 5.5 + j11 = 12.3\angle 63.43^0\,A.$$

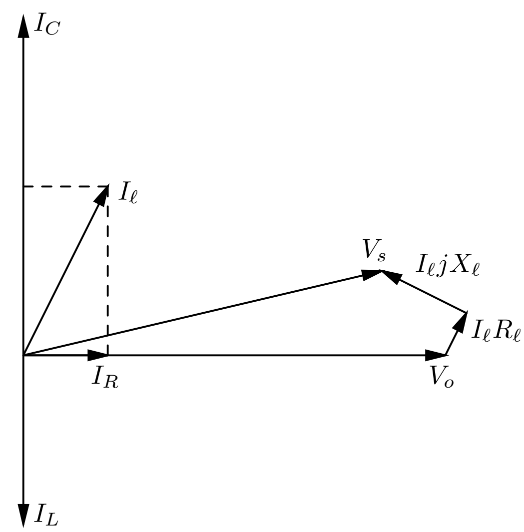
The voltage drop across R_ℓ equals $I_\ell R_\ell = 5.5 + j11 = 12.3\angle 63.43^0\,V$.

The voltage drop across jX_ℓ equals $I_\ell jX_\ell = -22 + j11 = 24.6\angle 153.43^0\,V$.

The total voltage drop in the line equals $5.5 + j11 - 22 + j11 = -16.5 + j22\,V$.

The supply voltage V_s is then given by

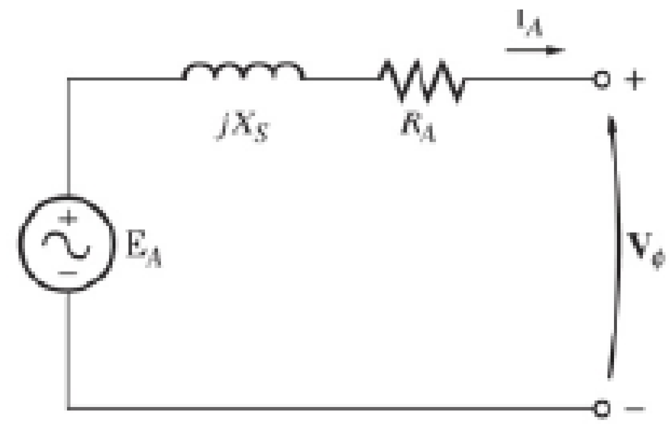
$$110 - 16.5 + j22 = 93.5 + j22 = 96.05\angle 13.24^0\,V.$$



The phasor diagram is shown above. Note that the scale for current phasors is different from that of voltage phasors.

Synchronous Generator Example

equivalent circuit



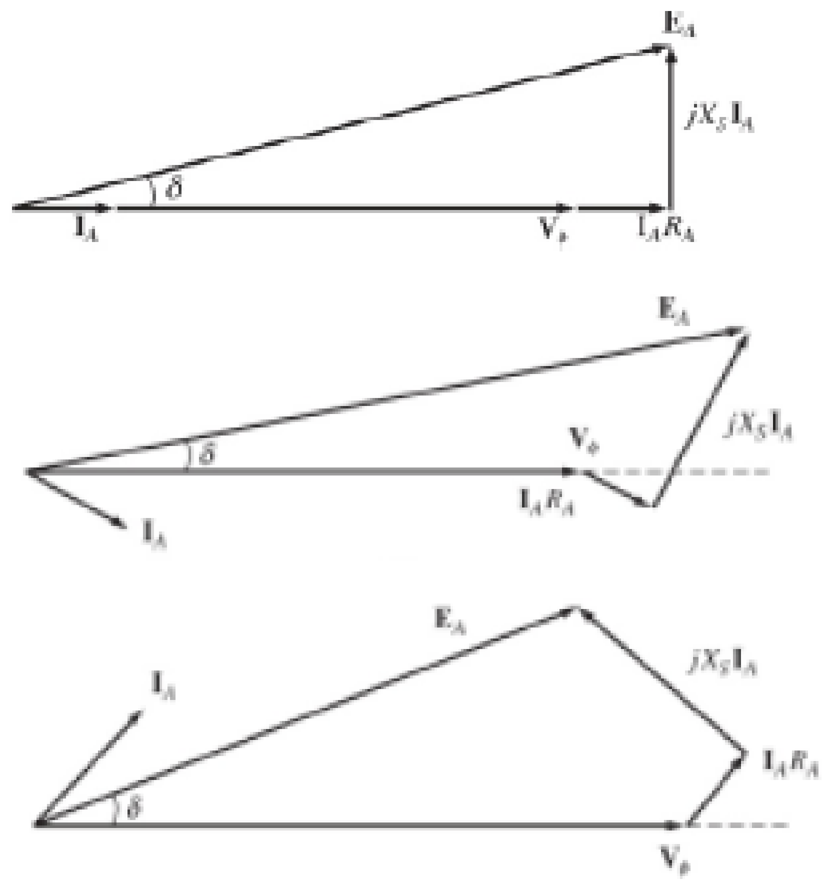
E_A is the generated voltage

V_ϕ is the terminal voltage across the load

I_A is the current supplied to the load

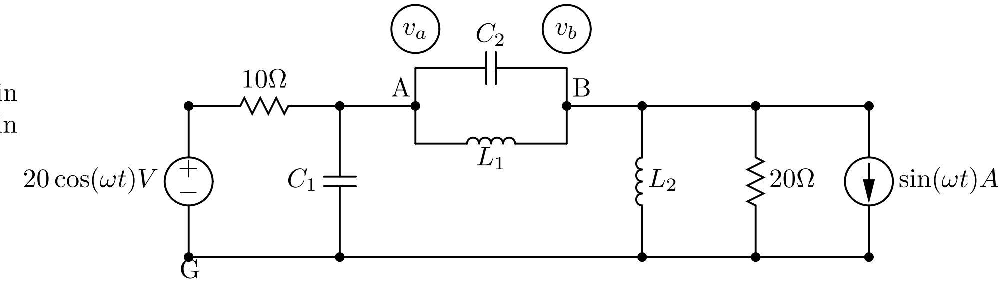
$$V_\phi = E_A - I_A R_A - jI_A X_s$$

$$E_A = V_\phi + I_A R_A + jI_A X_s$$

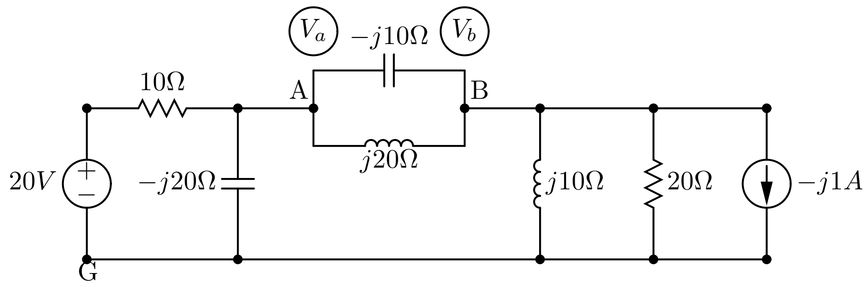


Example

Consider the time domain circuit shown on the right in which $\omega = 1000$, $C_1 = 50\mu F$, $C_2 = 100\mu F$, $L_1 = 20mH$, and $L_2 = 10mH$. Determine $v_a - v_b$.



The phasor equivalent of a circuit is as shown. Using $\cos(\omega t)$ as the reference signal, the source signal $20\cos(\omega t)$ transforms as a phasor 20 V, while $\sin(\omega t) = \cos(\omega t - 90^0)$ transforms as $-j1$ A. Also, we have the impedances $j\omega L_1 = j20\Omega$, $j\omega L_2 = j10\Omega$, $\frac{1}{j\omega C_1} = -j20\Omega$, and $\frac{1}{j\omega C_2} = -j10\Omega$.



Our aim is to determine $V_a - V_b$. This can be done by a number of methods. Just to illustrate several aspects of circuit analysis, we will first simplify the circuit by several steps and then determine $V_a - V_b$.

Simplification of parallel combination of $-j10$ and $j20$: The parallel equivalent of $-j10$ and $j20$ is given by

$$\frac{(-j10)(j20)}{-j10 + j20} = \frac{(-j^2 200)}{j10} = -j20.$$

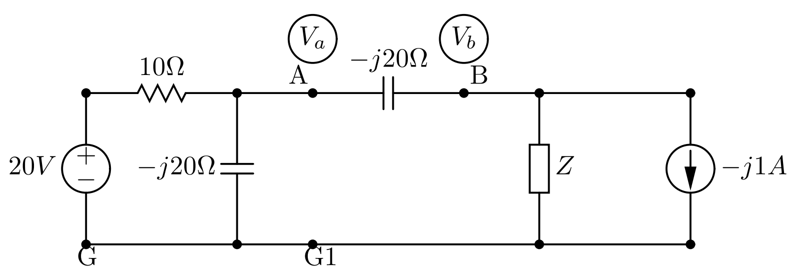
Simplification of parallel combination of $j10$ and 20: The parallel equivalent of $j10$ and 20 is given by

$$Z = \frac{(j10)(20)}{20 + j10} = \frac{(j200)}{20 + j10} = \frac{j20}{2 + j} = \frac{(j20)(2 - j)}{(2 + j)(2 - j)} = \frac{(j40 - j^2 20)}{2^2 + 1^2} = \frac{20(1 + j2)}{5} = 4(1 + j2)$$

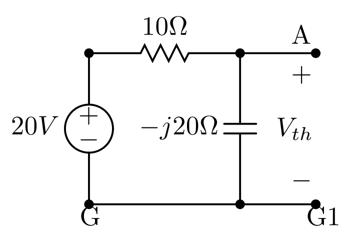
$$= \frac{20\angle 90^0}{\sqrt{5}\angle 26.57^0} = 4\sqrt{5}\angle 63.43 = 4(1 + j2).$$

We could have added admittance of each element in parallel, and then invert the resulting admittance to get the impedance.

With the above simplifications, the given circuit can be redrawn as shown on the right.

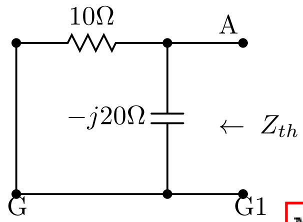


We will consider next only a part of the above circuit and simplify it. Consider the circuit to the left of terminals G1 and A as shown on the left side diagram, and construct its Thevenin equivalent. By voltage division rule, we can easily compute



$$V_{th} = \frac{20(-j20)}{10 - j20} = \frac{-j40}{1 - j2} = \frac{-j40(1 + j2)}{(1 - j2)(1 + j2)} = \frac{(-j40 - j^2 80)}{5} = 16 - j8$$

$$= \frac{40\angle -90^0}{\sqrt{5}\angle -63.43^0} = 8\sqrt{5}\angle -26.57^0 = 16 - j8.$$



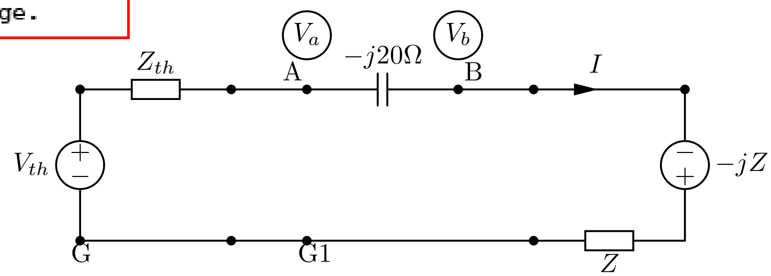
We can also easily compute the Thevenin impedance Z_{th} as the parallel combination of 10 and $-j20$,

$$Z_{th} = \frac{10(-j20)}{10-j20} = \frac{-j20}{1-j2} = \frac{(-j20)(1+j2)}{5} = \frac{-j20-j^2 40}{5} = 8-j4$$

$$= \frac{20\angle -90^0}{\sqrt{5}\angle -63.43^0} = 4\sqrt{5}\angle -26.57^0 = 8-j4.$$

Algebra is similar to what we did on the top of previous page.

We can also convert the current source in parallel with an impedance into a voltage source in series with an impedance. Thus, we can redraw equivalently the given circuit as shown on the right. We can write the loop equation as



$$V_{th} - jZ - I(Z_{th} + Z - j20) = 0 \quad \Rightarrow \quad I = \frac{V_{th} - jZ}{Z_{th} + Z - j20}.$$

We can now determine $V_a - V_b = I(-j20) = \frac{V_{th}-jZ}{Z_{th}+Z-j20}(-j20)$. Substituting the numbers for various variables, and simplifying we get

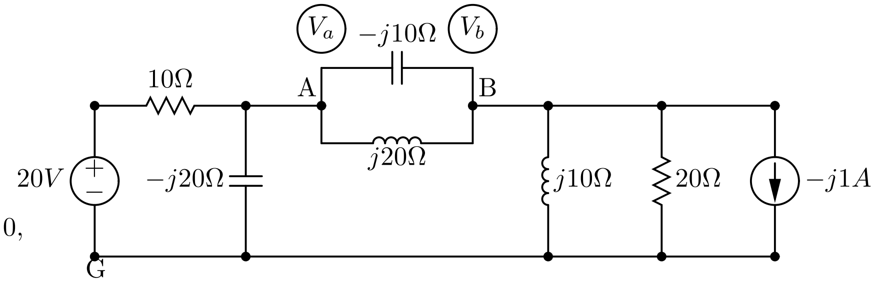
$$V_a - V_b = \frac{16-j8-j4(1+j2)}{8-j4+4(1+j2)-j20}(-j20) = \frac{(24-j12)(-j20)}{12-j16}$$

$$= \frac{12(2-j1)(-j5)}{3-j4} = \frac{12\sqrt{5}\angle -26.56^0 (5\angle -90^0)}{5\angle -53.13^0} = 12\sqrt{5}\angle -63.43^0 = 12(1-j2).$$

We can now go back from phasor domain to time domain,

$$v_a - v_b = 12\sqrt{5}\cos(1000t - 63.43^0).$$

Node Voltage Method:



For this example, node voltage method is much easier than any other method. We can write the node equations at A and B as follows:

$$\frac{V_a-20}{10}+\frac{V_a}{-j20}+\frac{V_a-V_b}{-j10}+\frac{V_a-V_b}{j20}=0,$$

$$-j1+\frac{V_b}{20}+\frac{V_b}{j10}+\frac{V_b-V_a}{-j10}+\frac{V_b-V_a}{j20}=0.$$

Multiplying the first equation by $j20$ and simplifying we get

$$j2V_a-j40-V_a-2V_a+2V_b+V_a-V_b=0 \;\Rightarrow\; -2(1-j)V_a+V_b=j40. \tag{1}$$

Similarly, multiplying the second equation by $j20$ and simplifying we get

$$20+jV_b+2V_b-2V_b+2V_a+V_b-V_a=0 \;\Rightarrow\; V_a+(1+j)V_b=-20. \tag{2}$$

Equations (1) and (2) are in a form suitable for any scientific calculator to solve them. We can solve them for V_a and V_b and then for V_a-V_b. If no scientific calculator is available, we can proceed to solve equations (1) and (2) simultaneously as follows: Multiplying equation (2) by $2(1-j)$ and adding to equation (1), we get

$$2(1-j)(1+j)V_b+V_b=-40(1-j)+j40 \;\Rightarrow\; 4V_b+V_b=-40+j80 \;\Rightarrow\; V_b=-8+j16.$$

Substituting for $V_b=-8+j16$ in equation (1) and simplifying, we get

$$-2(1-j)V_a-8+j16=j40 \;\Rightarrow\; V_a=\frac{8+j24}{-2(1-j)}=\frac{(8+j24)(1+j)}{-2(2)}=-2(1+j3)(1+j)=4-j8.$$

Once we know $V_a=4-j8$ and $V_b=-8+j16$, we can get

$$V_a-V_b=12(1-j2)=12\sqrt{5}\underline{/-63.}43^{0}.$$

We can now go back from phasor domain to time domain,

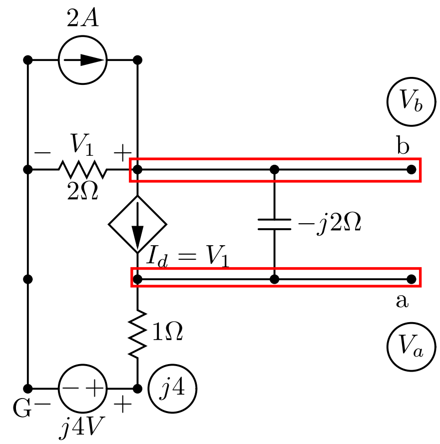
$$v_a-v_b=12\sqrt{5}\cos(1000t-63.43^{0}).$$

Mesh Current Method: We can also solve this circuit by Mesh Current Method, however we will face **five** mesh currents out of which only one is known and four unknown. So we need to write **four** equations and solve them simultaneously. This presents a lot more complex algebra. You can try it just to practice complex algebra.

Thevenin Equivalent Circuit – Example

The phasor equivalent of a circuit is shown. We would like to determine the Thevenin equivalent circuit at the terminals a and b. Note that the dependent current source I_d has a value equal to the voltage V_1. To do so, we plan to compute the open circuit voltage, the short circuit current, and Z_{Th} by using a test source. (Only two of these three items are required; for verification we will compute all three of them.)

Determination of V_{oc} by Node voltage method: Using G as the reference, we mark the node voltages as shown. Here V_b and V_a are unknown. We need to express the controlling voltage V_1 in terms of the node voltages. We see easily that $V_1 = V_b$. Since we have two unknowns, we need to write two node equations. The KCL equation at node b gives us



$$-2 + \frac{V_b}{2} + V_b + \frac{V_b - V_a}{-j2} = 0.$$

This equation simplifies to

$$j4 - jV_b - j2V_b + V_b - V_a = 0 \quad \Rightarrow \quad V_a = j4 + (1 - j3)V_b.$$

We can write another KCL equation either at node a or at node G. The KCL equation at node G (using super node concept) gives us

$$\frac{j4 - V_a}{1} + \frac{0 - V_b}{2} + 2 = 0 \quad \Rightarrow \quad V_b + 2V_a = 4(1 + j2).$$

Substituting $V_a = j4 + (1 - j3)V_b$, and simplifying, we get

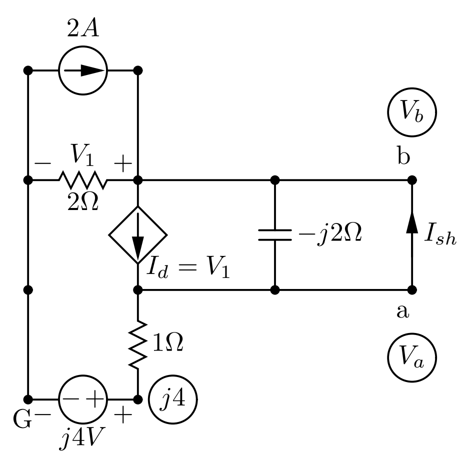
$$V_b + 2(j4 + (1 - j3)V_b) = 4(1 + j2) \quad \Rightarrow \quad V_b = \frac{4}{3 - j6} = \frac{4(3 + j6)}{45} = \frac{4(1 + j2)}{15} = 0.26666 + j0.53333.$$

Then

$$V_a = j4 + (1 - j3)V_b = \frac{28 + j56}{15} = 1.86666 + j3.73333.$$

The open circuit voltage $V_{oc} = V_a - V_b = 1.6 + j3.2$ V with terminal a at a higher potential than terminal b.

Determination of I_{sh} by Node voltage method: Using G as the reference, we mark the node voltages as shown. We note that $V_b = V_a = V_1$. There are two unknowns, V_1 and I_{sh}. We write two KCL node equations one at the node a and the other at the node b. The KCL at the node a is given by



$$I_{sh} - V_1 + \frac{V_1 - j4}{1} = 0 \quad \Rightarrow \quad I_{sh} = j4.$$

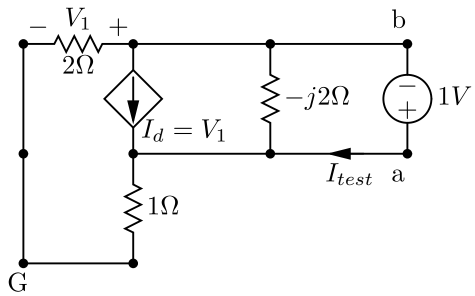
We do not need to write the KCL equation at node b, unless we want to determine V_1. Writing the equation, we get

$$-2 + \frac{V_1}{2} + V_1 - I_{sh} = 0 \quad \Rightarrow \quad 3V_1 = 4 + 2I_{sh} \quad \Rightarrow \quad V_1 = \frac{4 + j8}{3}.$$

Thus

$$Z_{Th} = \frac{V_{oc}}{I_{sh}} = \frac{1.6 + j3.2}{j4} = 0.8 - j0.4\Omega.$$

Determination of Z_{Th} by using a test source: The independent sources are set to zero, i.e the current source is opened and the voltage source is shorted. The resulting circuit is shown on the right where a test voltage of 1 V between the terminals b and a is applied. It is easy to compute V_1 as $-\frac{2}{3}$V . Knowing V_1, we can easily compute I_{test} as

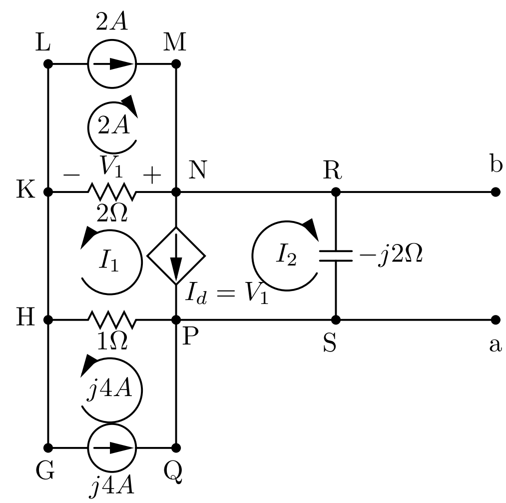


$$I_{test} = \frac{1}{-j2} + \frac{1}{3} + \frac{2}{3} = 1 + j0.5A.$$

Thus

$$Z_{Th} = \frac{1}{1+j0.5} = 0.8 - j0.4\Omega.$$

Determination of V_{oc} by Mesh current method: Just for illustration, we used source transformation, and redrew the given circuit as shown while marking all the mesh currents in it. Here I_1 and I_2 are unknown. We need to express the controlling voltage V_1 in terms of the mesh currents. We see easily that controlling voltage $V_1 = 2(I_1 + 2)$. The dependent current source I_d has a value equal to the voltage V_1. On the other hand, in terms of the chosen mesh currents $I_d = -(I_1 + I_2)$. Thus, we get the relationship,



$$I_d = -(I_1 + I_2) = V_1 = 2(I_1 + 2) \;\Rightarrow\; -(I_1 + I_2) = 2(I_1 + 2).$$

We need one more equation. The KVL equation HKNRSPH gives us

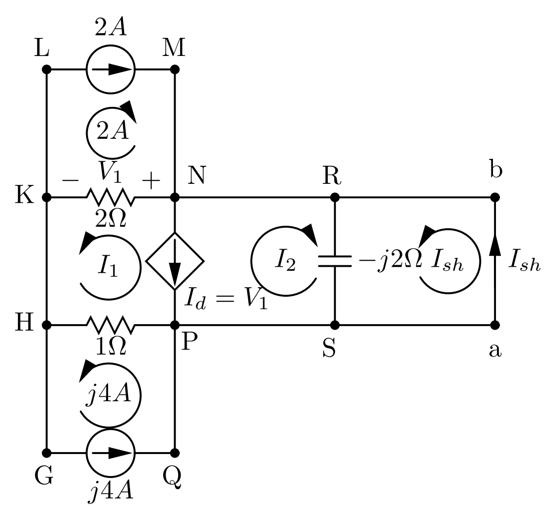
$$2(I_1 + 2) + j2I_2 - (j4 - I_1) = 0.$$

By solving the above equations, we get

$$I_1 = \frac{-5.6 + j0.8}{3} \quad \text{and} \quad I_2 = 1.6 - j0.8.$$

Thus, the open circuit voltage $V_{oc} = j2I_2 = 1.6 + j3.2$V with the terminal a being at a higher potential than the terminal b.

Determination of I_{sh} by Mesh current method: Just for illustration, we used source transformation, and redrew the given circuit as shown while marking all the mesh currents in it. Here I_1, I_2, and I_{sh} are unknown. We need to express the controlling voltage V_1 in terms of the mesh currents. We see easily that $V_1 = 2(I_1 + 2)$. The dependent current source V_1 can be related to the mesh currents as



$$V_1 = 2(I_1 + 2) = -(I_1 + I_2).$$

We need two more equations. The KVL equation HKNRbaSPH gives us

$$2(I_1 + 2) - (j4 - I_1) = 0.$$

The KVL equation RbaSR gives us

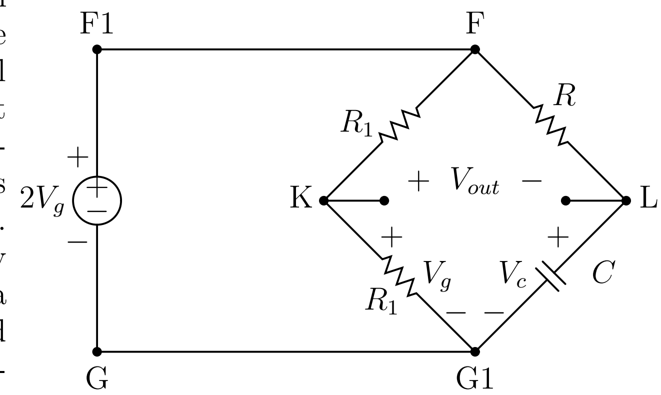
$$-j2(I_2 + I_{sh}) = 0.$$

Solving the above equations, we get $3I_1 = -4 + j4$ A, $I_2 = -I_{sh}$, and $I_{sh} = j4$ A.

332:221 Principles of Electrical Engineering I

Phase Shifting Circuit

Many applications require a phase shift of an incoming sinusoidal signal. This can be done by a **Phase Shifting Circuit**. One typical phase shifting circuit is shown on the right where $2V_g$ denotes the phasor of incoming sinusoidal signal of frequency ω and V_{out} denotes the phasor of a phase shifted sinusoidal signal. The amount of phase shift can be varied by varying R or C but typically R is chosen as a potentiometer whose resistance can be tuned as desired. In the circuit, R_1 is a fixed resistance.



By voltage division rule, it is easy to note that the voltage across each resistance R_1 is V_g. Similarly, by voltage division rule, the voltage across the capacitance is given by

$$V_c = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} 2V_g = \frac{1}{1 + j\omega CR} 2V_g = \frac{1}{\sqrt{1+\omega^2C^2R^2}\angle\theta} 2V_g = \frac{1}{\sqrt{1+\omega^2C^2R^2}} 2V_g\angle -\theta$$

where $\theta = \tan^{-1}(\omega CR)$. Note that a phase shift is explicitly seen in V_c, however amplitude of V_c depends on R and C. By defining V_{out} as $V_g - V_c$, as seen below, we can render the amplitude of V_{out} independent of R and C.

We can determine the output voltage as $\boxed{1 + j\omega CR - 2}$

$$V_{out} = V_g - V_c = V_g - \frac{1}{1+j\omega CR} 2V_g = \left[\frac{1}{2} - \frac{1}{1+j\omega CR}\right] 2V_g = \frac{-1 + j\omega CR}{1 + j\omega CR} V_g.$$

We can further simplify the expression for V_{out},

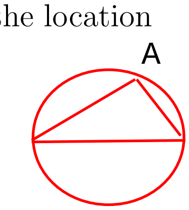
$$V_{out} = \frac{\sqrt{1+\omega^2C^2R^2}\angle 180^{\circ} - \theta}{\sqrt{1+\omega^2C^2R^2}\angle\theta} V_g = V_g\angle 180^{\circ} - 2\theta \quad \text{where } \theta = \tan^{-1}(\omega CR).$$

This indicates that the output amplitude is half of input amplitude and does not depend on R and C. Also, more importantly the output has a phase shift of $180^{\circ} - 2\theta$ compared to the input. Note that the phase angle shift can be varied by varying either R or C.

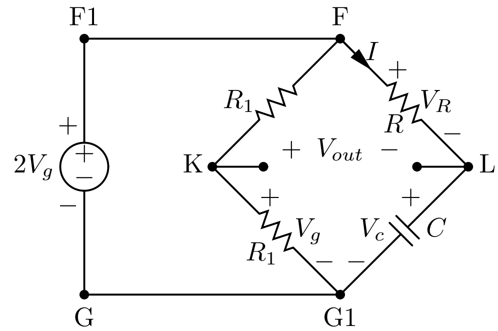
Although it is a little more involved than the above simple analysis, a graphical phasor analysis can be done to get a better picture of phase shift. This is shown on the next page.

A property of a circle: The phasor analysis shown on the next page depends on one fundamental property of a circle:

- Consider any diameter of a circle. Draw a segment from one end of the diameter to meet the circumference at some point, say A. Draw another segment from the other end of the diameter to meet the circumference at the some point A. Then, the two segments drawn are perpendicular to each other at the circumference irrespective of the location of point A as long as it lies on the circumference.



Phase Shifting Circuit – Graphical Analysis for fun



Let us re-draw the Phase Shifting Circuit as shown by marking the voltage across R and current through it. We note that

$$\begin{aligned} I &= \frac{1}{R+\frac{1}{j\omega C}}2V_g = \frac{j\omega C}{1+j\omega CR}2V_g \\ &= \frac{\omega C\angle 90^0}{\sqrt{1+\omega^2C^2R^2}\angle\theta}2V_g = \frac{\omega C}{\sqrt{1+\omega^2C^2R^2}}\,2V_g\angle 90^0-\theta \end{aligned}$$

where $\theta = \tan^{-1}(\omega CR)$. Also,

$$\boxed{V_R = IR} = \frac{\omega CR}{\sqrt{1+\omega^2C^2R^2}}\,2V_g\angle 90^0-\theta$$

As expected, I and V_R are in phase, however they lead $2V_g$ by $90^0-\theta$. The voltage V_R is represented in the following figure by the phasor GA.
As determined previously,

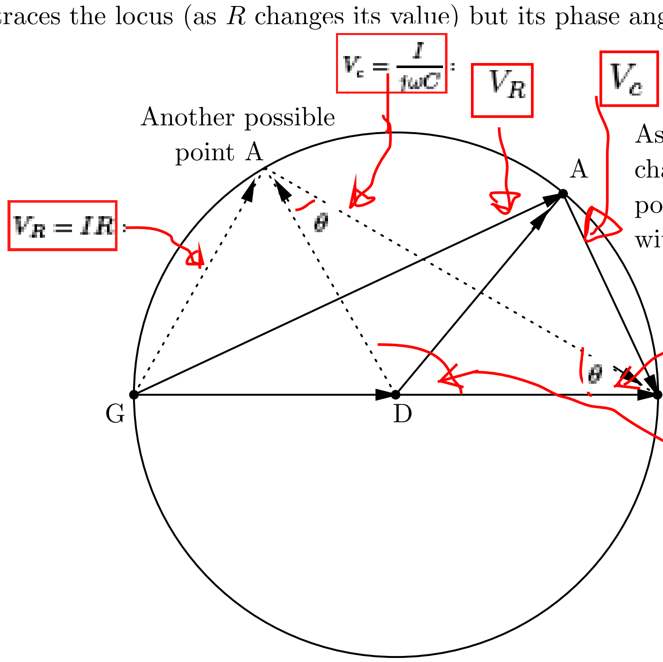
$$\boxed{V_c = \frac{I}{j\omega C}} = \frac{1}{\sqrt{1+\omega^2C^2R^2}}\,2V_g\angle -\theta$$

Clearly, $\boxed{V_c \text{ lags } V_R \text{ by exactly } 90^0}$ whatever are the values of R and C. This fact has an important role as discussed soon. The voltage V_c is represented in the following figure by the phasor AB.
Note that

$$\boxed{V_R + V_c = 2V_g.}$$

The voltage $2V_g$ is represented in the following figure by the phasor GB. Also, the phasor GD represents V_g with D being the mid point of GB.

As the value of R changes, the magnitudes of V_R and V_c change and hence the location of point A changes in the figure. What is the locus of point A as the value of R changes? The 90^0 phase lag between V_c and V_R has an important consequence in determining the locus. In fact, the locus is a semi-circle, this is because the segment GB is fixed and because the angle between GA and AD is always 90^0 irrespective of the value of R.
For small R, the magnitude of V_R is small and the magnitude of V_c is large, and thus the point A on the semi-circle locus is near by G. On the other hand, for large R, the magnitude of V_R is large and the magnitude of V_c is small, and thus the point A on the semi-circle locus is near by D.
We note that the phasor DA equals DB+BA=DB-AB. However, DB represents V_g and AB represents V_c. Thus, DA=DB-AB represents $V_g - V_c$ which is V_{out}. This means DA represents V_{out}. From the properties of triangles, the phase angle of DA is $\angle 180^0 - 2\theta$. Since DA is the radius of the semi-circle its length is fixed as A traces the locus (as R changes its value) but its phase angle can be tuned by tuning R.



As R changes, the location of A changes. The locus of all possible points of A is the semi-circle with center D and radius DA.

$\boxed{\theta = \tan^{-1}(\omega CR)}$
Angle ABD is θ
Angle DAB is θ
Angle AGD is $90^o - \theta$
Angle GAD is $90^o - \theta$
Angle ADB is $\boxed{180^o - 2\theta}$

DA represents V_{out}.
Note that Angle ADB $= 180^o - 2\theta$ is the phase angle of V_{out}.
The magnitude of V_{out} is always V_g.

Transfer Function of a Circuit

Let us first emphasize the concept of impedance in Laplace domain and in Phasor domain:

All electrical engineering signals exist in time domain where time t is the independent variable. One can **transform a time-domain signal to phasor domain for sinusoidal signals**.
For general signals not necessarily sinusoidal, one can **transform a time domain signal into a Laplace domain signal**.

$$\text{The impedance of an element in Laplace domain} = \frac{\text{Laplace Transform of its voltage}}{\text{Laplace Transform of its current}}.$$

$$\text{The impedance of an element in phasor domain} = \frac{\text{Phasor of its voltage}}{\text{Phasor of its current}}.$$

The impedances of elements, R, L, and C are given by

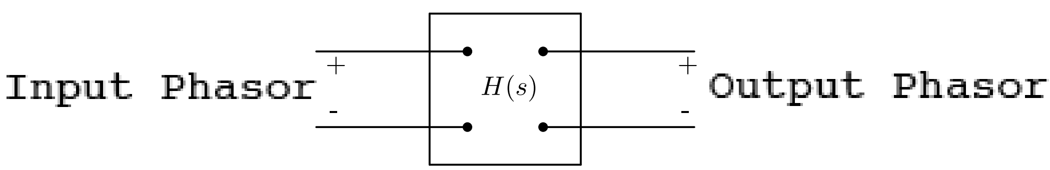
Element :	Resistance **R**	Inductance **L**	Capacitance **C**
Impedance in Laplace domain :	$\mathbf{R}$	$\mathbf{sL}$	$\frac{\mathbf{1}}{\mathbf{sC}}$
Impedance in Phasor domain :	$\mathbf{R}$	$\mathbf{j\omega L}$	$\frac{\mathbf{1}}{\mathbf{j\omega C}}$

$$\boxed{s = j\omega}$$

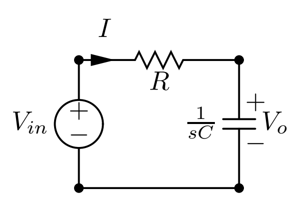
For Phasor domain, the Laplace variable $\boxed{s = j\omega}$ where ω is the radian frequency of the sinusoidal signal.

The transfer function $H(s)$ **of a circuit is defined as**:

$$\boxed{H(s) = \text{The transfer function of a circuit} \;=\; \frac{\text{Transform of the output}}{\text{Transform of the input}} = \frac{\text{Phasor of the output}}{\text{Phasor of the input}}.}$$



Example: As a simple example, consider a RC circuit as shown on the right. By voltage division rule, it is easy to determine its transfer function as

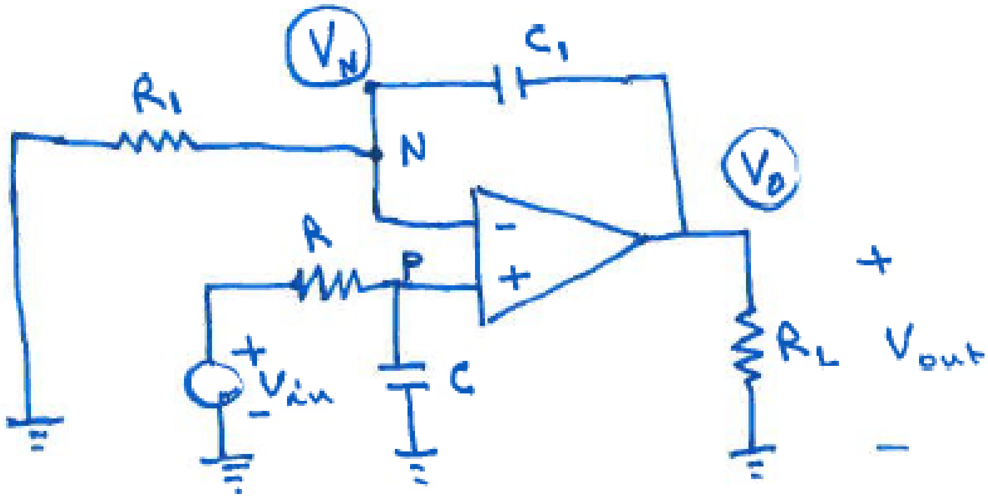


$$H(s) = \frac{V_o}{V_{in}} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC} = \frac{1}{RC}\,\frac{1}{s + \frac{1}{RC}} = \frac{\alpha}{s + \alpha}$$

where $\alpha = \frac{1}{RC}$.

Transfer function is normally expressed in a form where the coefficient of highest power in the denominator is unity (one).

Example: Determine the transfer function of the circuit shown. Assume that the Op-Amp is ideal.



The solution is simple. In what follows we show all steps clearly showing all the mathematical manipulations.

By voltage division rule,

$$V_N = V_P = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_{in} = \frac{1}{1 + sRC} V_{in} = \frac{1}{RC} \frac{1}{s + \frac{1}{RC}} V_{in} = \frac{\alpha}{s + \alpha} V_{in}$$

where $\alpha = \frac{1}{RC}$.

We can write the node equation at N as

$$(V_N - V_0)sC_1 + \frac{V_N}{R_1} = 0.$$

We can simplify the above equation as

$$V_N - V_0 + \frac{V_N}{sC_1R_1} = 0 \quad \Rightarrow \quad V_0 = V_N + \frac{V_N}{sC_1R_1} = V_N \left[1 + \frac{1}{sC_1R_1} \right] = V_N \frac{1 + sC_1R_1}{sC_1R_1}.$$

Thus

$$V_0 = V_N \frac{1 + sC_1R_1}{sC_1R_1} = V_N \frac{s + \frac{1}{C_1R_1}}{s} = V_N \frac{s + \beta}{s}$$

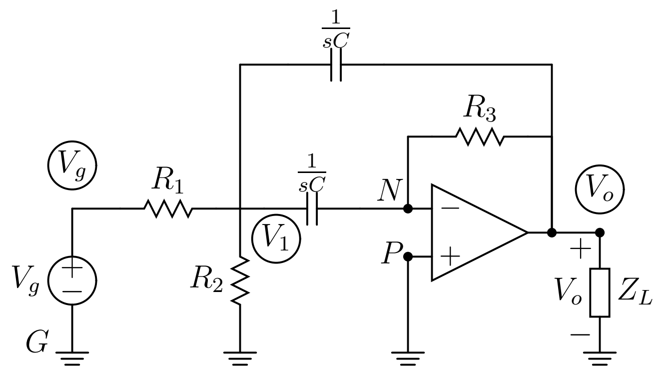
where $\beta = \frac{1}{C_1R_1}$.

We get

$$\text{The transfer function } = H(s) = \frac{V_0}{V_{in}} = \frac{V_0}{V_N} \frac{V_N}{V_{in}} = \frac{s + \beta}{s} \, \frac{\alpha}{s + \alpha} = \frac{\alpha(s + \beta)}{s(s + \alpha)}.$$

This is often used in deriving filter circuits.

Transfer Function of a BPF Op-Amp Circuit



Determine the transfer function of the ideal Op-Amp circuit shown where the output is V_o and the input is V_g. Express it in the form

$$H(s) = \frac{-K\beta s}{s^2 + \beta s + \omega_o^2}$$

and identify the values of K, β, and ω_o^2 in terms of the circuit parameters R_1, R_2, R_3, and C.

Let G be the reference node, and consider the node voltages as marked. Then the node equation at node V_1 can be written as

$$\frac{V_1 - V_g}{R_1} + sCV_1 + \frac{V_1}{R_2} + sC(V_1 - V_o) = 0.$$

By re-writing, we get

$$\left[2sC + \frac{1}{R_1} + \frac{1}{R_2}\right] V_1 - sCV_o = \frac{V_g}{R_1}.$$

We can write a second node equation at N as

$$\frac{-V_o}{R_3} - sCV_1 = 0 \quad \Rightarrow \quad V_1 = -\frac{1}{sCR_3}V_o.$$

Substituting the above in the very first node equation, we get

$$-\left[\frac{2}{R_3} + \frac{1}{sCR_1R_3} + \frac{1}{sCR_2R_3} + sC\right] V_o = \frac{V_g}{R_1}.$$

Multiplying throughout by $-\frac{s}{C}$ and simplifying, we get

$$\left[s^2 + \frac{2s}{R_3C} + \frac{1}{C^2R_3}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right] V_o = -\frac{sV_g}{R_1C}.$$

Let

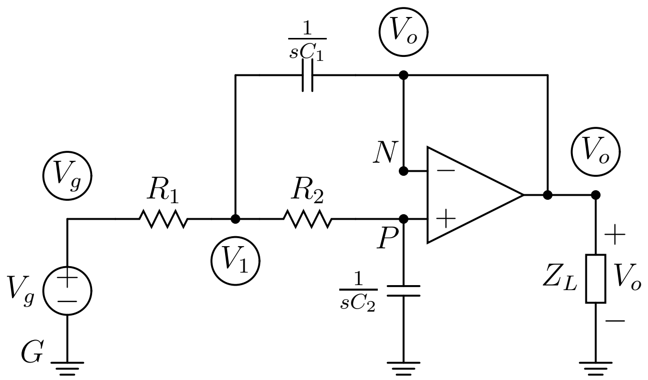
$$\omega_o^2 = \frac{1}{C^2R_3}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad \beta = \frac{2}{R_3C},$$

$$K = \frac{R_3}{2R_1} \ \text{ so that } \ K\beta = \frac{1}{R_1C}.$$

Then the transfer function is given by

$$\frac{V_o}{V_g} = H(s) = \frac{-K\beta s}{s^2 + \beta s + \omega_o^2}.$$

Home Work, not collected; related problems will be given in the upcoming exam.



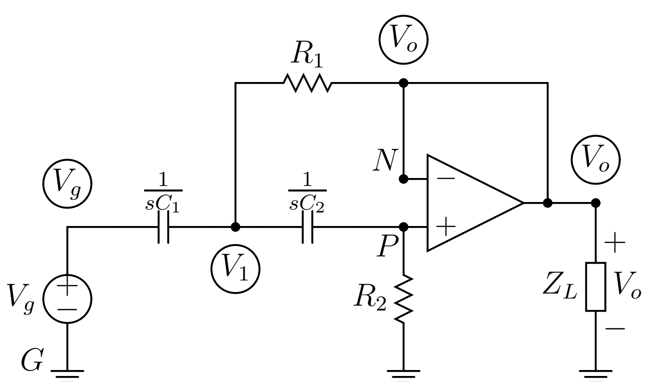
Determine the transfer function of the ideal Op-Amp circuit shown where the output is V_o and the input is V_g. Express it in the form

$$H(s) = \frac{K}{s^2 + \beta s + \omega_o^2}.$$

To avoid excessive algebra, assume that $R_1 = R_2 = R$ and $C_1 = C_2 = C$.

Answer:

$$K = \omega_o^2 = \frac{1}{R^2C^2} \quad \text{and} \quad \beta = \frac{2}{RC}.$$



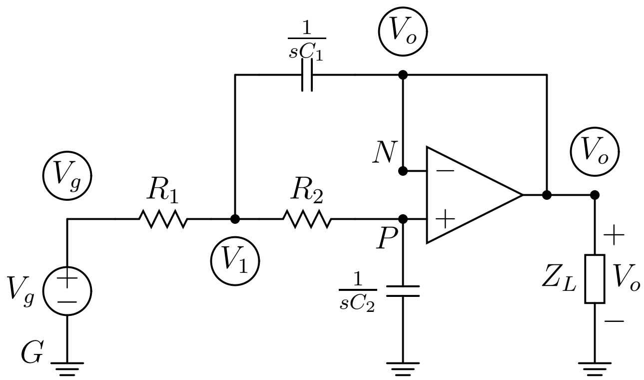
Determine the transfer function of the ideal Op-Amp circuit shown where the output is V_o and the input is V_g. Express it in the form

$$H(s) = \frac{Ks^2}{s^2 + \beta s + \omega_o^2}.$$

To avoid excessive algebra, assume that $R_1 = R_2 = R$ and $C_1 = C_2 = C$.

Answer:

$$K = 1, \;\; \omega_o^2 = \frac{1}{R^2C^2} \quad \text{and} \quad \beta = \frac{2}{RC}.$$



Determine the transfer function of the ideal Op-Amp circuit shown where the output is V_o and the input is V_g. Express it in the form

$$H(s) = \frac{K}{s^2 + \beta s + \omega_o^2}$$

and identify the values of K, β, and ω_o^2 in terms of the circuit parameters R_1, R_2, C_1, and C_2.

Let G be the reference node, and consider the node voltages as marked. Then the node equation at node V_1 can be written as

$$sC_1(V_1 - V_o) + \frac{V_1 - V_g}{R_1} + \frac{V_1 - V_o}{R_2} = 0.$$

We can write a second node equation at P as

$$\frac{V_o - V_1}{R_2} + sC_2V_o = 0.$$

From the above equation, we get

$$V_1 - V_o = sR_2C_2\,V_o \;\text{ and }\; V_1 = (1 + sR_2C_2)V_o.$$

Substituting the above in the very first node equation, we get

$$\left[sC_1\,sR_2C_2 + \frac{1 + sR_2C_2}{R_1} + \frac{sR_2C_2}{R_2}\right]V_o = \frac{V_g}{R_1}.$$

Dividing throughout by $C_1R_2C_2$ and simplifying, we get

$$\left[s^2 + \frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)s + \frac{1}{R_1R_2C_1C_2}\right]V_o = \frac{V_g}{R_1R_2C_1C_2}.$$

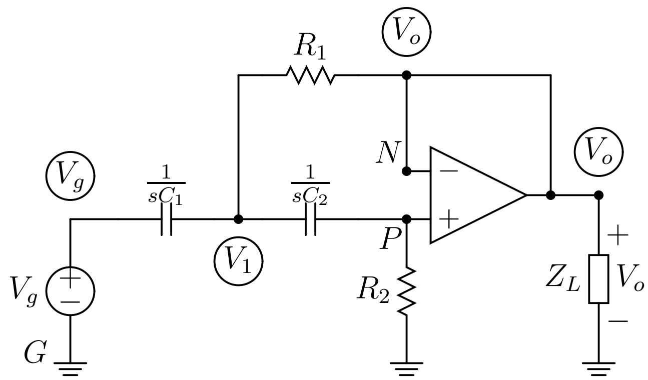
Let

$$K = \omega_o^2 = \frac{1}{R_1R_2C_1C_2} \;\text{ and }\; \beta = \frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

Then the transfer function is given by

$$\frac{V_o}{V_g} = H(s) = \frac{\omega_o^2}{s^2 + \beta s + \omega_o^2}.$$

Transfer Function of a HPF Op-Amp Circuit



Determine the transfer function of the ideal Op-Amp circuit shown where the output is V_o and the input is V_g. Express it in the form

$$H(s) = \frac{Ks^2}{s^2 + \beta s + \omega_o^2}$$

and identify the values of K, β, and ω_o^2 in terms of the circuit parameters R_1, R_2, C_1, and C_2.

Let G be the reference node, and consider the node voltages as marked. Then the node equation at node V_1 can be written as

$$sC_1(V_1 - V_g) + \frac{V_1 - V_o}{R_1} + sC_2(V_1 - V_o) = 0.$$

We can write a second node equation at P as

$$\frac{V_o}{R_2} + sC_2(V_o - V_1) = 0.$$

From the above equation, we get

$$V_1 - V_o = \frac{1}{sR_2C_2}V_o \quad \text{and} \quad V_1 = \frac{1 + sR_2C_2}{sR_2C_2}V_o.$$

Substituting the above in the very first node equation, we get

$$\left[sC_1\frac{1 + sR_2C_2}{sR_2C_2} + \frac{1}{sR_1R_2C_2} + sC_2\frac{1}{sR_2C_2}\right] V_o = sC_1V_g.$$

We can re-write this equation as

$$\left[\frac{C_1}{R_2C_2} + sC_1 + \frac{1}{sR_1R_2C_2} + \frac{1}{R_2}\right] V_o = sC_1V_g.$$

Multiplying throughout by $\frac{s}{C_1}$, and simplifying, we get

$$\left[s^2 + \frac{1}{R_2}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)s + \frac{1}{R_1R_2C_1C_2}\right] V_o = s^2V_g.$$

Let

$$K = 1, \;\; \omega_o^2 = \frac{1}{R_1R_2C_1C_2} \quad \text{and} \quad \beta = \frac{1}{R_2}\left(\frac{1}{C_1} + \frac{1}{C_2}\right).$$

Then the transfer function is given by

$$\frac{V_o}{V_g} = H(s) = \frac{s^2}{s^2 + \beta s + \omega_o^2}.$$

HW from Nilsson and Riedel 8th and 9th editions

Some of these problems will appear as quiz or exam problems.

Nilsson and Riedel 8th edition:

9.2, 9.11, 9.12, 9.13, 9.15, 9.23, 9.29, 9.40, 9.48, 9.53, 9.60

Nilsson and Riedel 9th edition:

9.5, 9.11, 9.13, 9.14, 9.16, 9.27, 9.33, 9.45, 9.48, 9.57, 9.63

Name in CAPITAL LETTERS: **HW: Addition of sinusoids in time domain**

Review of Trigonometry – Read this page, HW on next page

Let $y_1(t) = A_1\cos(\omega t + \theta_1)$ and $y_2(t) = A_2\cos(\omega t + \theta_2)$. Let $y(t) = y_1(t) + y_2(t)$. Write $y(t)$ in the form $y(t) = A\cos(\omega t + \theta)$ and find values for A and θ in terms of A_1, A_2, θ_1 and θ_2.

$$\begin{aligned}
y(t) &= y_1(t) + y_2(t)\\
&= A_1\cos(\omega t + \theta_1) + A_2\cos(\omega t + \theta_2)\\
&= A_1\cos(\theta_1)\cos(\omega t) - A_1\sin(\theta_1)\sin(\omega t) + A_2\cos(\theta_2)\cos(\omega t) - A_2\sin(\theta_2)\sin(\omega t)\\
&= [A_1\cos(\theta_1) + A_2\cos(\theta_2)]\cos(\omega t) - [A_1\sin(\theta_1) + A_2\sin(\theta_2)]\sin(\omega t)\\
&= A\cos(\theta)\cos(\omega t) - A\sin(\theta)\sin(\omega t)
\end{aligned}$$

We note that

$$A\cos(\theta) = A_1\cos(\theta_1) + A_2\cos(\theta_2) \quad \text{and} \quad A\sin(\theta) = A_1\sin(\theta_1) + A_2\sin(\theta_2).$$

Thus

$$\begin{aligned}
A &= \sqrt{A^2\cos^2(\theta) + A^2\sin^2(\theta)}\\
&= \sqrt{[A_1\cos(\theta_1) + A_2\cos(\theta_2)]^2 + [A_1\sin(\theta_1) + A_2\sin(\theta_2)]^2},
\end{aligned}$$

$$\begin{aligned}
\tan(\theta) &= \frac{A\sin(\theta)}{A\cos(\theta)}\\
&= \frac{A_1\sin(\theta_1) + A_2\sin(\theta_2)}{A_1\cos(\theta_1) + A_2\cos(\theta_2)}.
\end{aligned}$$

Example : Let $y_1(t) = 20\cos(\omega t - 30^o)$ and $y_2(t) = 40\cos(\omega t + 60^o)$. Then

$$\begin{aligned}
y(t) &= y_1(t) + y_2(t)\\
&= 20\cos(\omega t - 30^o) + 40\cos(\omega t + 60^o)\\
&= 20\cos(-30^o)\cos(\omega t) - 20\sin(-30^o)\sin(\omega t) + 40\cos(60^o)\cos(\omega t) - 40\sin(60^o)\sin(\omega t)\\
&= [20\cos(-30^o) + 40\cos(60^o)]\cos(\omega t) - [20\sin(-30^o) + 40\sin(60^o)]\sin(\omega t)\\
&= A\cos(\theta)\cos(\omega t) - A\sin(\theta)\sin(\omega t)
\end{aligned}$$

We note that

$$A\cos(\theta) = 20\cos(-30^o) + 40\cos(60^o) = 37.32 \text{ and}$$

$$A\sin(\theta) = 20\sin(-30^o) + 40\sin(60^o) = 24.64.$$

Thus

$$\begin{aligned}
A &= \sqrt{A^2\cos^2(\theta) + A^2\sin^2(\theta)}\\
&= \sqrt{37.32^2 + (-24.64)^2} = 44.72,
\end{aligned}$$

and

$$\begin{aligned}
\tan(\theta) &= \frac{A\sin(\theta)}{A\cos(\theta)}\\
&= \frac{20\sin(-30^o) + 40\sin(60^o)}{20\cos(-30^o) + 40\cos(60^o)} = 0.66.
\end{aligned}$$

Thus $\theta = 33.43^o$, and hence

$$y(t) = 44.72\cos(\omega t + 33.43^o).$$

It is very important to note that, in the algebra done on the left, only the amplitude and phase of the sinusoidal signals play a role; the time domain signals do not explicitly come into the algebra of adding two sinusoids.

Review trigonometric expressions given below:

$$
\begin{aligned}
\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\
\cos(\alpha) &= \sin(\alpha + 90^{0}) \ \text{ and } \ \sin(\alpha) = \cos(\alpha - 90^{0}) \\
\sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \ \text{ and } \ 1 = \cos^2(\alpha) + \sin^2(\alpha) \\
2\cos^2(\alpha) &= 1 + \cos(2\alpha) \ \text{ and } \ 2\sin^2(\alpha) = 1 - \cos(2\alpha)
\end{aligned}
$$

Example to be done as Home-Work

Consider the addition of two sinusoidal signals

$$5\cos(100t + 36.9^{o}) + 5\cos(100t + 53.1^{o}).$$

Express the above sum in the form $A\cos(100t + \theta)$, and determine the numerical values for A and θ. **Show all your algebra**.
If you do your algebra correctly, you should get $A = 9.9$ and $\theta = 45^{o}$.

Name in CAPITAL LETTERS: **HW: Complex Algebra**
LAST FOUR DIGITS OF ID NUMBER:

Complex Numbers – Home-Work

Example 1: Determine the polar form of the following four numbers:
$a = 1.5321 + j1.2856$
$b = -1.5321 + j1.2856$
$c = -1.5321 - j1.2856$
$d = 1.5321 - j1.2856$
Hint: Since all these four numbers in rectangular form are distinct from one another, the corresponding polar forms must be distinct from one another as well.

Example 2: Let $a = 4+j3$ and $b = 1-j$. Determine $a+b$, $a-b$, ab and $\frac{a}{b}$ both in rectangular and polar forms. *Answers in the order:* $5 + j2 = 5.385\underline{/21.8^0}$, $3 + j4 = 5\underline{/53.13}^0$, $7 - j1 = 5\sqrt{2}\underline{/-8.1}^0$, $0.5 + j3.5 = \frac{5}{\sqrt{2}}\underline{/81.9^0}$.

Example 3: Let $a = j2$, $b = 2 - j2$, and $c = 1 + j5$. Determine $\frac{2a}{b+c}$ both in rectangular and polar forms. *Answer:* $\frac{2}{3}(1 + j1) = \frac{2\sqrt{2}}{3}\underline{/45^0.}$

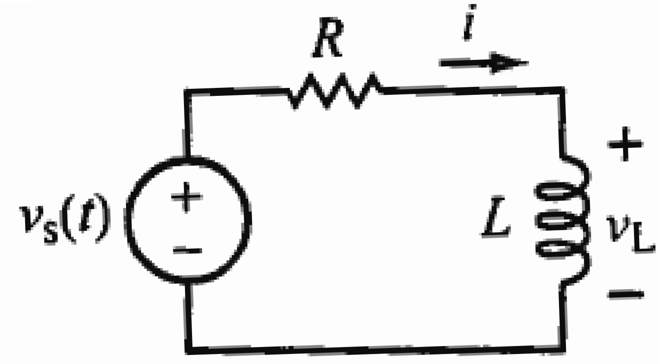
Example 4: Let $a = 2 - j3$, $b = 6 + j2$, $c = 2 - j3$, and $d = 6 + j1$. Determine $\frac{ab}{c+d}$ both in rectangular and polar forms. *Answer:* $\frac{43}{17} - j\frac{19}{17} = 2.5294 - j1.117 = 2.7653\underline{/-23.}839^0$.

RL Circuit – Phasor Domain Analysis

In the RL Circuit shown, determine the voltage $v_L(t)$ across the inductance when $R = 3\Omega$, $L = 10^{-4}$ H, and

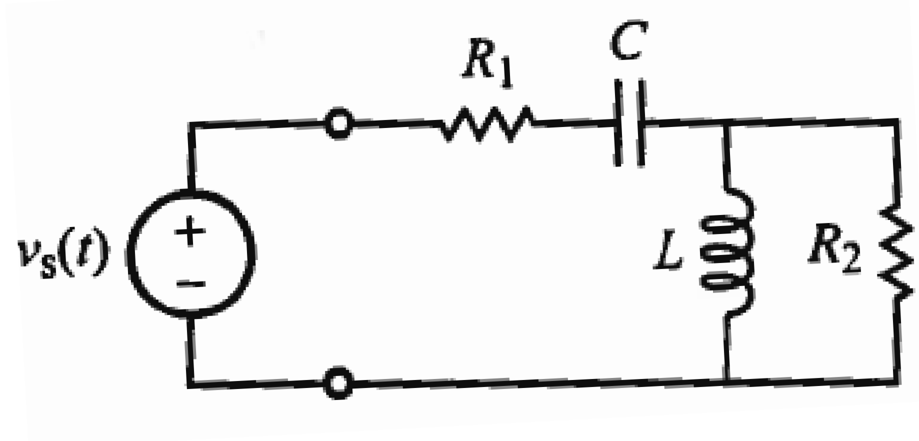
$v_s(t) = 15\cos(4{\times}10^4 t{-}60^{\circ})$.

Transform the given time domain circuit into phasor domain.
Solve for the phasor domain voltage V_L across the inductance.
Transform V_L in to time domain to get $v_L(t)$.



Time domain

Determine the input impedance (that is the impedance seen by the input) if $\omega = 10^6$ rad/sec, $R_1 = 2\ \text{k}\Omega$, $R_2 = 4\ \text{k}\Omega$, $L = 3$ mH, and $C = 1$ nF. Note carefully the notations k, m, and n.



Name in CAPITAL LETTERS: **HW: AC Node Voltage Method**

LAST FOUR DIGITS OF ID NUMBER:

332:221 Principles of Electrical Engineering I

This is a HW problem that is collected and graded

Part 1:
Consider the circuit shown in Figure 1a where the independent current source i_g and i_s are-given by

$$i_g = 4\cos(4t), \quad i_s = 2\cos(4t).$$

Note that the frequency of the sources equals 4 radians/sec. Our aim is to determine i_1 when the circuit is in sinusoidal steady state. In order to do the analysis, we plan to use **node voltage method** with node g as the reference node. We divide the analysis into several steps. In this problem, both algebra as well as concepts are important. Check your algebra carefully in each step.
No other method of solving for the current i_1 is acceptable.

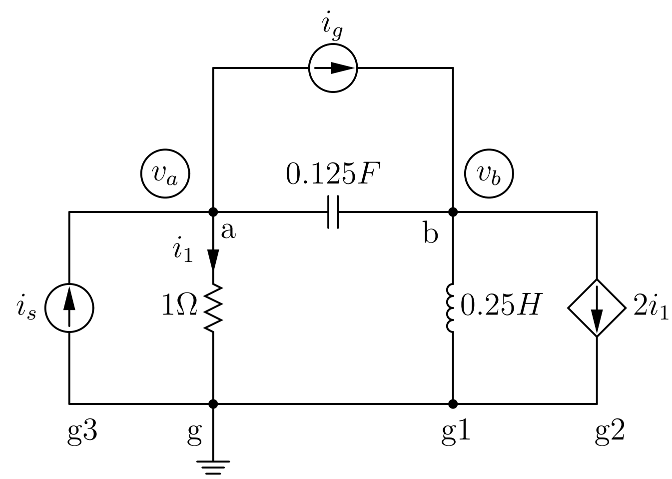


Figure 1a

Step 1, Conversion of time domain circuit into phasor domain: The phasor domain circuit can be drawn as in Figure 1b. Determine the values of I_s, I_g, Z_1, Z_2, and Z_3 and mark them in Figure 1b. In determining I_s and I_g, use the amplitude as the magnitude of the phasor.

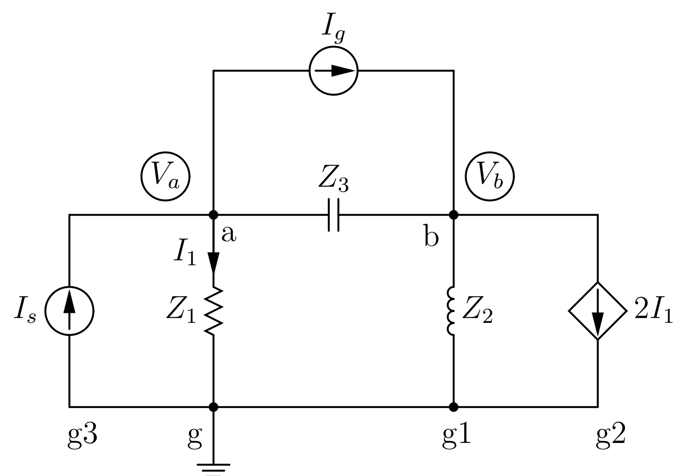


Figure 1b

Step 2: Write the KCL at node 'a'.

Step 3: Write the KCL at node 'b' .

Step 4, algebra: *Check the above equations carefully once again. If any of the above equations is incorrect, no credit can be given for that equation as well as the following algebra.* Some one who correctly solved the above two equations informed us that

$$V_b = 9 - j7.$$

Determine at first V_a.

Determine next I_1.

Transform the phasor I_1 into time domain, that is write a time domain expression for i_1.

Name in CAPITAL LETTERS: **HW: AC Mesh Current Method**

LAST FOUR DIGITS OF ID NUMBER:

332:221 Principles of Electrical Engineering I

This is a HW problem that is collected and graded

Consider the circuit shown in Figure 1a where the independent current sources i_g and i_s are given by

$$i_g = 4\cos(4t), \quad i_s = 2\cos(4t).$$

Note that the frequency of the sources equals 4 radians/sec. Our aim is to determine i_1 when the circuit is in sinusoidal steady state. In order to do the analysis, we plan to use **mesh current method**. We divide the analysis into several steps. In this problem, both algebra as well as concepts are important. Check your algebra carefully in each step.
No other method of solving for the current i_1 is acceptable.

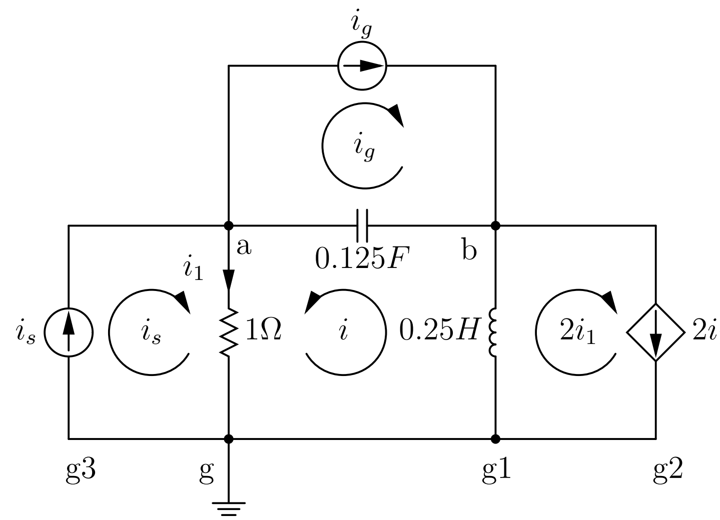


Figure 1a

Step 1, Conversion of time domain circuit into phasor domain: The phasor domain circuit can be drawn as in Figure 1b. Determine the values of I_s, I_g, Z_1, Z_2, and Z_3 and mark them in Figure 1b. In determining I_s and I_g, use the amplitude as the magnitude of the phasor.

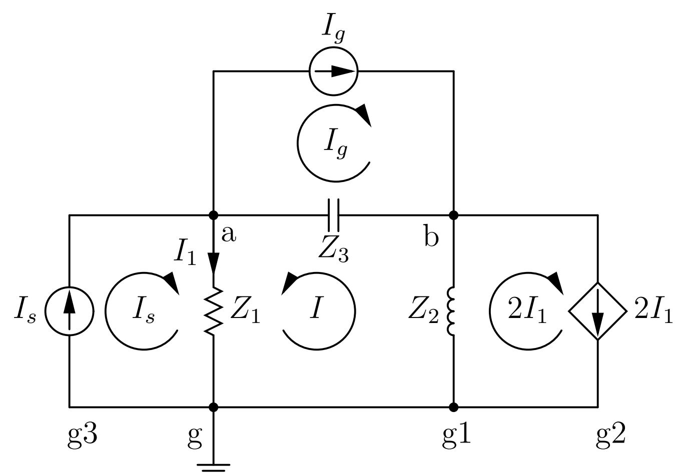


Figure 1b

Step 2: Use the mesh currents as shown. Write the mesh equation (KVL) for the loop 'gabg1g'.

Step 3, algebra: *Check the above equation carefully once again. If it is correct,*
$I = (1 + j3)$.
Determine the value of I_1.

Transform the phasor I_1 into time domain, that is write a time domain expression for i_1.

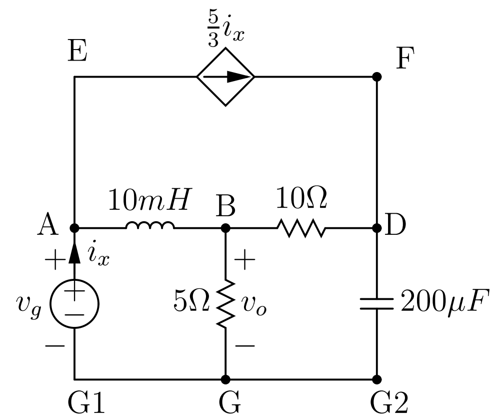
Name in CAPITAL LETTERS: **HW: AC Three Problems**

LAST FOUR DIGITS OF ID NUMBER:

Principles of Electrical Engineering I

Sinusoidal steady state analysis – Home-Work

Bridged-T-Example 1:
Transform the AC time-domain circuit shown into phasor domain for the purpose of sinusoidal steady state analysis. Do the phasor-analysis, and then determine the output voltage $v_o(t)$ across $5\,\Omega$. You need to write the phasor-domain equations by both Node Voltage and Mesh Current Methods. However, you can complete the analysis by doing the algebra utilizing the equations of either method. (*Answer: The phasor domain voltage* $V_o = 10\angle 45^0$. *Other answers are on the last page.*)

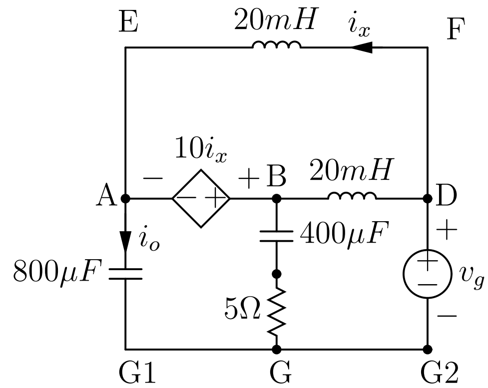


$v_g(t) = 20\cos(500t + 45^0)$

Node voltage method: Select your own node voltage variables and mark them on the circuit. Write down all the node equations that arise by Node voltage method below.

Mesh current method: Select your own mesh current variables and mark them on the circuit. Write down all the mesh equations that arise by Mesh current method below.

Bridged-T-Example 2:
Transform the AC time-domain circuit shown into phasor domain for the purpose of sinusoidal steady state analysis. Do the phasor-analysis, and then determine the phasor voltage and phasor current of each circuit element. You need to write the phasor-domain equations by both Node Voltage and Mesh Current Methods. However, you can complete the analysis by doing the algebra utilizing the equations of either method. (*Partial answer: The phasor domain current* I_o *through* $800\,\mu F$ *is* $2 + j2 = 2\sqrt{2}\angle 45^o$.
Other answers are on the last page.)



$v_g(t) = 10\cos(250t)$

Node voltage method: Select your own node voltage variables and mark them on the circuit. Write down all the node equations that arise by Node voltage method below.

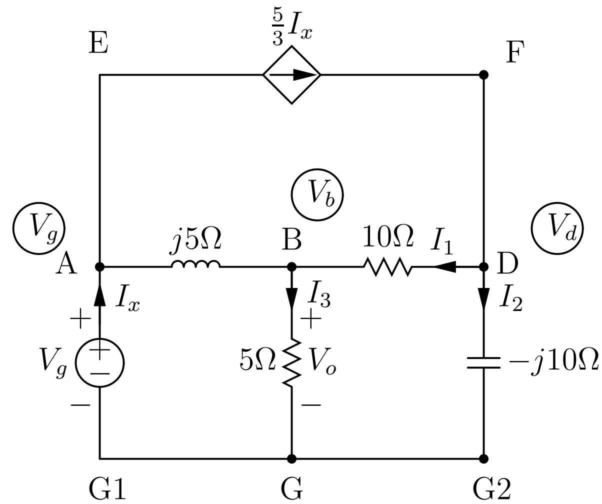
Mesh current method: Select your own mesh current variables and mark them on the circuit. Write down all the mesh equations that arise by Mesh current method below.

Example 3:

Consider the circuit of Example 2, and determine the real and reactive power consumed or generated by each element by utilizing the voltages and currents obtained in Example 2. There should be power balance within the expected numerical accuracies.

Use the voltage and current variables given on the next page for Example 2.

Example 1 Answers:
Let G be the reference node, then
$V_a = V_g = 20\angle 45^0 = 10\sqrt{2}(1+j)$ V
$V_b = V_o = 10\angle 45^0 = 5\sqrt{2}(1+j)$ V
$V_d = \frac{10+j50}{\sqrt{2}} = 5\sqrt{2}(1+j5)$ V
$I_1 = j2\sqrt{2}$ A
$I_2 = \frac{-5+j}{\sqrt{2}}$ A
$I_3 = \frac{2+j2}{\sqrt{2}} = \sqrt{2}(1+j)$ A
$I_x = \frac{-3+j3}{\sqrt{2}}$ A



Example 2 Answers:
Let G be the reference node, then
$V_a = 10\sqrt{2}\angle -45^0 = 10(1-j)$ V
$V_b = 10(3-j)$ V
$V_d = V_g = 10$ V
$I_1 = 2(1+j2)$ A
$I_2 = -4(1+j)$ A
$I_3 = 2(1+j)$ A
$I_x = 2$ A
$I_o = 2(1+j) = 2\sqrt{2}\angle 45^0$ A