Transfer Function of a Circuit

Let us first emphasize the concept of impedance in Laplace domain and in Phasor domain:

All electrical engineering signals exist in time domain where time $t$ is the independent variable. One can transform a time-domain signal to phasor domain for sinusoidal signals.

For general signals not necessarily sinusoidal, one can transform a time domain signal into a Laplace domain signal.

The impedance of an element in Laplace domain $= \frac{\text{Laplace Transform of its voltage}}{\text{Laplace Transform of its current}}$.

The impedance of an element in phasor domain $= \frac{\text{Phasor of its voltage}}{\text{Phasor of its current}}$.

The impedances of elements, $R$, $L$, and $C$ are given by

<table>
<thead>
<tr>
<th>Element</th>
<th>Resistance $R$</th>
<th>Inductance $L$</th>
<th>Capacitance $C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impedance in Laplace domain</td>
<td>$R$</td>
<td>$sL$</td>
<td>$\frac{1}{sC}$</td>
</tr>
<tr>
<td>Impedance in Phasor domain</td>
<td>$R$</td>
<td>$j\omega L$</td>
<td>$\frac{1}{j\omega C}$</td>
</tr>
</tbody>
</table>

For Phasor domain, the Laplace variable $s = j\omega$ where $\omega$ is the radian frequency of the sinusoidal signal.

The transfer function $H(s)$ of a circuit is defined as:

$H(s) = \frac{\text{Transform of the output}}{\text{Transform of the input}} = \frac{\text{Phasor of the output}}{\text{Phasor of the input}}$

Example: As a simple example, consider a RC circuit as shown on the right. By voltage division rule, it is easy to determine its transfer function as

$$H(s) = \frac{V_o}{V_{in}} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC} = \frac{1}{RC} \frac{1}{\frac{1}{RC} + \frac{1}{sC}} = \frac{\alpha}{s + \alpha}$$

where $\alpha = \frac{1}{RC}$.

Transfer function is normally expressed in a form where the coefficient of highest power in the denominator is unity (one).
Example: Determine the transfer function of the circuit shown. Assume that the Op-Amp is ideal.

By voltage division rule,

\[ V_N = V_P = \frac{1}{sC} \frac{V_{in}}{R + \frac{1}{sC}} V_{in} = \frac{1}{1 + sRC} V_{in} = \frac{1}{sRC} \left( \frac{1}{sRC} + V_{in} \right) = \frac{\alpha}{s + \alpha} V_{in} \]

where \( \alpha = \frac{1}{RC} \).

We can write the node equation at N as

\[ (V_N - V_0) sC_1 + \frac{V_N}{R_1} = 0. \]

We can simplify the above equation as

\[ V_N - V_0 + \frac{V_N}{sC_1 R_1} = 0 \Rightarrow V_0 = V_N + \frac{V_N}{sC_1 R_1} = V_N \left[ 1 + \frac{1}{sC_1 R_1} \right] = V_N \frac{1 + sC_1 R_1}{sC_1 R_1}. \]

Thus

\[ V_0 = V_N \frac{1 + sC_1 R_1}{sC_1 R_1} = V_N \frac{s + \frac{1}{C_1 R_1}}{s} = V_N \frac{s + \beta}{s} \]

where \( \beta = \frac{1}{C_1 R_1} \).

We get

The transfer function \( H(s) = \frac{V_0}{V_{in}} = \frac{V_0}{V_N} \frac{V_N}{V_{in}} = \frac{s + \beta}{s} \frac{1}{s + \alpha} = \frac{\alpha(s + \beta)}{s(s + \alpha)}. \)

This is often used in deriving filter circuits.